Contents 9

Vectors

Learning outcomes

In this Workbook you will learn what a vector is and how to combine vectors together using the triangle law. You will be able to represent a vector by its Cartesian components. You will be able to multiply vectors together using either the scalar product or the vector product. You will be able to apply your knowledge of vectors to solve problems involving forces and to geometric problems involving lines and planes.

Basic Concepts of Vectors

9.1

Introduction

In engineering, frequent reference is made to physical quantities, such as force, speed and time. For example, we talk of the speed of a car, and the force in a compressed spring. It is useful to separate these physical quantities into two types. Quantities of the first type are known as **scalars**. These can be fully described by a single number known as the **magnitude**. Quantities of the second type are those which require the specification of a **direction**, in addition to a magnitude, before they are completely described. These are known as **vectors**. Special methods have been developed for handling vectors in calculations, giving rise to subjects such as vector algebra, vector geometry and vector calculus. Quantities that are vectors must be manipulated according to certain rules, which are described in this and subsequent Sections.

Prerequisites

Before starting this Section you should ...

- be familiar with all the basic rules of algebra

Learning Outcomes

On completion you should be able to ...

- categorize a number of common physical quantities as scalar or vector
- represent vectors by directed line segments
- combine, or add, vectors using the triangle law
- resolve a vector into two perpendicular components

1. Introduction

It is useful to separate physical quantities into two types: the first are called **scalars**; the second are known as **vectors**. A scalar is a quantity that can be described by a single number which can be positive, negative or zero. An example of a scalar quantity is the mass of an object, so we might state that 'the mass of the stone is 3 kg'. It is important to give the units in which the quantity is measured. Obvious examples of scalars are temperature and length, but there are many other engineering applications in which scalars play an important role. For example, speed, work, voltage and energy are all scalars.
On the other hand, vectors are quantities which require the specification of a **magnitude** and a **direction**. An example of a vector quantity is the force applied to an object to make it move. When the object shown in Figure 1 is moved by applying a force to it we achieve different effects depending on the direction of the force.

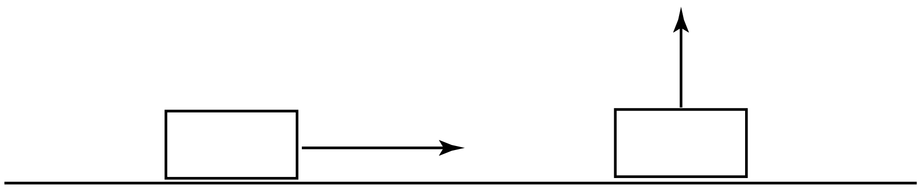


Figure 1: A force is a vector quantity

In order to specify the force completely we must state not only its magnitude (its 'strength') but also the direction in which the force acts. For example we might state that 'a force of 5 newtons is applied vertically from above'. Clearly this force would have a different effect from one applied horizontally. The *direction* in which the force acts is crucial.

There are many engineering applications where vectors are important. Force, acceleration, velocity, electric and magnetic fields are all described by vectors. Furthermore, when computer software is written to control the position of a robot, the position is described by vectors.

Sometimes confusion can arise because words used in general conversation have specific technical meanings when used in engineering calculations. An example is the use of the words 'speed' and 'velocity'. In everyday conversation these words have the same meaning and are used interchangeably. However in more precise language they are not the same. **Speed** is a scalar quantity described by giving a single number in appropriate units; for example 'the speed of the car is 40 kilometres per hour'. On the other hand **velocity** is a vector quantity and must be specified by giving a direction as well. For example 'the velocity of the aircraft is 20 metres per second due north'.
In engineering calculations, the words speed and velocity cannot be used interchangeably. Similar problems arise from use of the words 'mass' and 'weight'. **Mass** is a scalar which describes the amount of substance in an object. The unit of mass is the kilogramme. **Weight** is a vector, the direction of which is vertically downwards because weight arises from the action of gravity. The unit of weight is the newton. **Displacement** and **distance** are related quantities which can also cause confusion. Whereas distance is a scalar, displacement is 'directed distance', that is, distance together with a specified direction. So, referring to Figure 2, if an object is moved from point A to point B, we can state that the distance moved is 10 metres, but the displacement is 10 metres *in the direction from A to B*.

You will meet many other quantities in the course of your studies and it will be helpful to know which are vectors and which are scalars. Some common quantities and their type are listed in Table 1. The S.I. units in which these are measured are also shown.

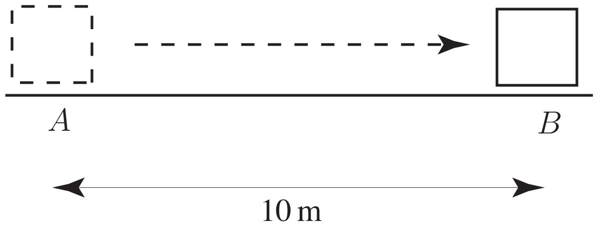


Figure 2: Displacement means directed distance

Table 1. Some common scalar and vector quantities

quantity	type	S.I. unit
distance	scalar	metre, m
mass	scalar	kilogramme, kg
temperature	scalar	kelvin, K
pressure	scalar	pascal, Pa
work	scalar	joule, J
energy	scalar	joule, J
displacement	vector	metre m
force	vector	newton, N
velocity	vector	metres per second, m s^{-1}
acceleration	vector	metres per second per second, m s^{-2}

Exercise

State which of the following are scalars and which are vectors:

(a) the volume of a petrol tank,
(b) a length measured in metres,
(c) a length measured in miles,
(d) the angular velocity of a flywheel,
(e) the relative velocity of two aircraft,
(f) the work done by a force,
(g) electrostatic potential,
(h) the momentum of an atomic particle.

Answer

(a), (b), (c) (f), (g) are scalars.

(d), (e), and (h) are vectors

2. The mathematical description of vector quantities

Because a vector has a direction as well as a magnitude we can represent a vector by drawing a line. The length of the line represents the **magnitude** of the vector given some appropriate scale, and the direction of the line represents the **direction** of the vector. We call this representation a **directed line segment**. For example, Figure 3 shows a vector which represents a velocity of 3 m s^{-1} north-west. Note that the arrow on the vector indicates the direction required.

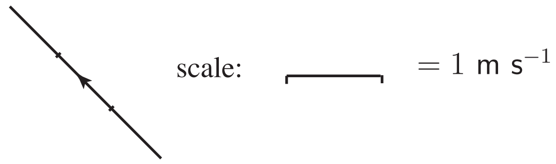


Figure 3: A vector quantity can be represented by drawing a line

More generally, Figure 4 shows an arbitrary vector quantity.

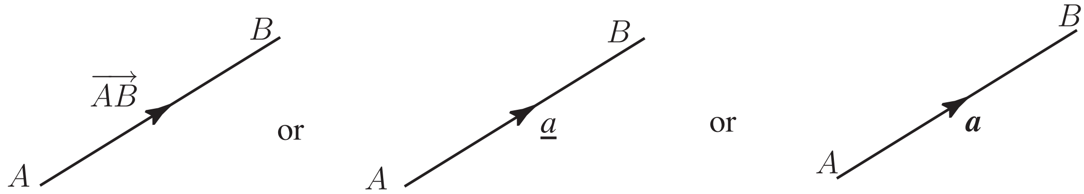


Figure 4: Representation of vectors

It is important when writing vectors to distinguish them from scalars. Various notations are used. In Figure 4 we emphasise that we are dealing with the vector from A to B by using an **arrow** and writing $\overrightarrow{AB}$. Often, in textbooks, vectors are indicated by using a **bold typeface** such as $\boldsymbol{a}$. It is difficult when handwriting to reproduce the bold face and so it is conventional to **underline** vector quantities and write $\underline{a}$ instead. So $\overrightarrow{AB}$ and $\underline{a}$ represent the same vector in Figure 4. We can also use the notation $\underline{AB}$. In general in this Workbook we will use underlining but we will also use the arrow notation where it is particularly helpful.

Example 1

Figure 5 shows an object pulled by a force of 10 N at an angle of $60°$ to the horizontal.

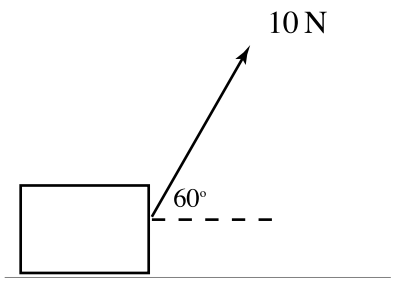


Figure 5

Show how this force can be represented by a vector.

Solution

The force can be represented by drawing a line of length 10 units at an angle of 60° to the horizontal, as shown below.

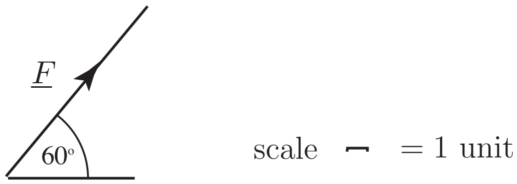


Figure 6

We have labelled the force $\underline{F}$. When several forces are involved they can be labelled $\underline{F}_1$, $\underline{F}_2$ and so on.

When we wish to refer simply to the magnitude (or length) of a vector we write this using the **modulus** sign as $|\overrightarrow{AB}|$, or $|\underline{a}|$, or simply a (without the underline.)

In general two vectors are said to be **equal vectors** if they have the same magnitude and same direction. So, in Figure 7 the vectors $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are equal even though their locations differ.

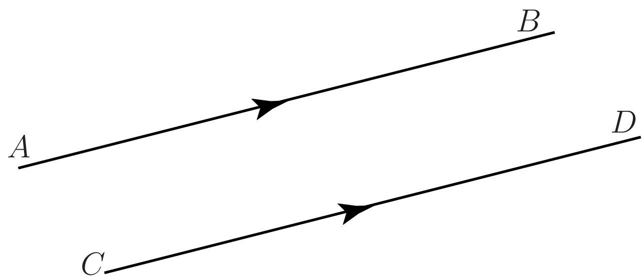


Figure 7: Equal vectors

This is a useful and important property of vectors: a vector is defined only by its direction and magnitude, not by its location in space. These vectors are often called **free** vectors.
The vector $-\underline{a}$ is a vector in the opposite direction to $\underline{a}$, but has the same magnitude as $\underline{a}$, as shown in Figure 8.

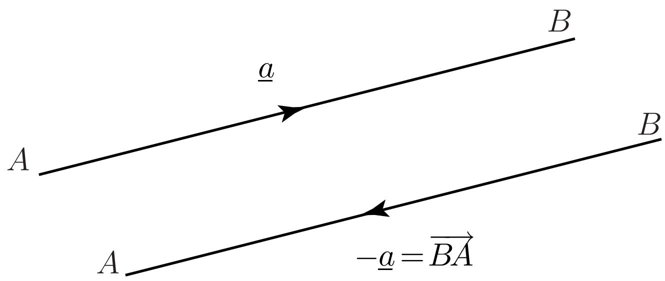


Figure 8

Geometrically, if $\underline{a} = \overrightarrow{AB}$ then $-\underline{a} = \overrightarrow{BA}$.

Exercises

1. An object is subject to two forces, one of 3 N vertically downwards, and one of 8 N, horizontally to the right. Draw a diagram representing these two forces as vectors.

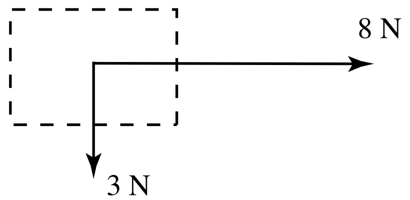
2. Draw a diagram showing an arbitrary vector $\underline{F}$. On the diagram show the vector $-\underline{F}$.

3. Vectors $\underline{p}$ and $\underline{q}$ are equal vectors. Draw a diagram to represent $\underline{p}$ and $\underline{q}$.

4. If $\underline{F}$ is a vector, what is meant by F?

Answers

1.



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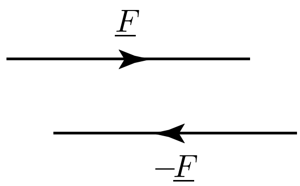
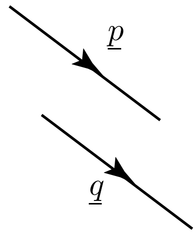


3.



4. F is the magnitude of $\underline{F}$.

3. Addition of vectors

Vectors are added in a particular way known as the **triangle law**. To see why this law is appropriate to add them this way consider the following example:

Example: The route taken by an automated vehicle

An unmanned vehicle moves on tracks around a factory floor carrying components from the store at A to workers at C as shown in Figure 9.

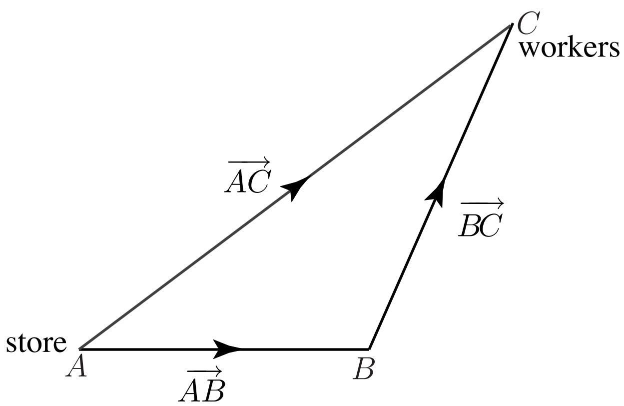


Figure 9

The vehicle may arrive at C either directly or via an intermediate point B. The movement from A to B can be represented by a displacement vector $\overrightarrow{AB}$. Similarly, movement from B to C can be represented by the displacement vector $\overrightarrow{BC}$, and movement from A to C can be represented by $\overrightarrow{AC}$. Since travelling from A to B and then B to C is equivalent to travelling directly from A to C we write

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

This is an example of the triangle law for adding vectors. We add vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ by placing the tail of $\overrightarrow{BC}$ at the head of $\overrightarrow{AB}$ and completing the third side of the triangle so formed ($\overrightarrow{AC}$).

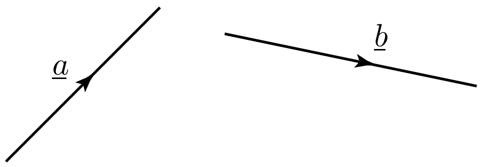


Figure 10: Two vectors $\underline{a}$ and $\underline{b}$

Consider the more general situation in Figure 10. Suppose we wish to add $\underline{b}$ to $\underline{a}$. To do this $\underline{b}$ is translated, keeping its direction and length unchanged, until its tail coincides with the head of $\underline{a}$. Then the sum $\underline{a} + \underline{b}$ is defined by the vector represented by the third side of the completed triangle, that is $\underline{c}$ in Figure 11. Note, from Figure 11, that we can write $\underline{c} = \underline{a} + \underline{b}$ since going along $\underline{a}$ and then along $\underline{b}$ is equivalent to going along $\underline{c}$.

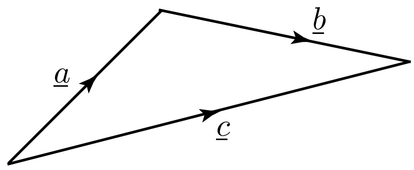
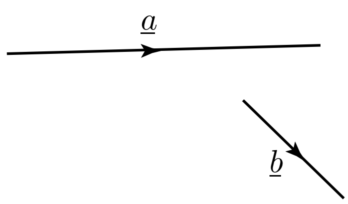


Figure 11: Addition of the two vectors of Figure 10 using the triangle law

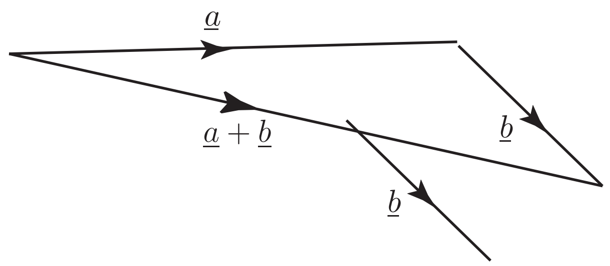
Task

Using vectors $\underline{a}$ and $\underline{b}$ shown below, draw a diagram to find $\underline{a}+\underline{b}$. Find also $\underline{b}+\underline{a}$.
Is $\underline{a} + \underline{b}$ the same as $\underline{b} + \underline{a}$?



Your solution

Answer



yes $\underline{a}+\underline{b}=\underline{b}+\underline{a}$

It is possible, using the triangle law, to prove the following rules which apply to any three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$:

Key Point 1

Vector Addition

$\underline{a}+\underline{b}=\underline{b}+\underline{a}$	vector addition is commutative
$\underline{a}+(\underline{b}+\underline{c})=(\underline{a}+\underline{b})+\underline{c}$	vector addition is associative

Example: Resultant of two forces acting upon a body

A force $\underline{F}_1$ of 2 N acts vertically downwards, and a force $\underline{F}_2$ of 3 N acts horizontally to the right, upon the body shown in Figure 12.

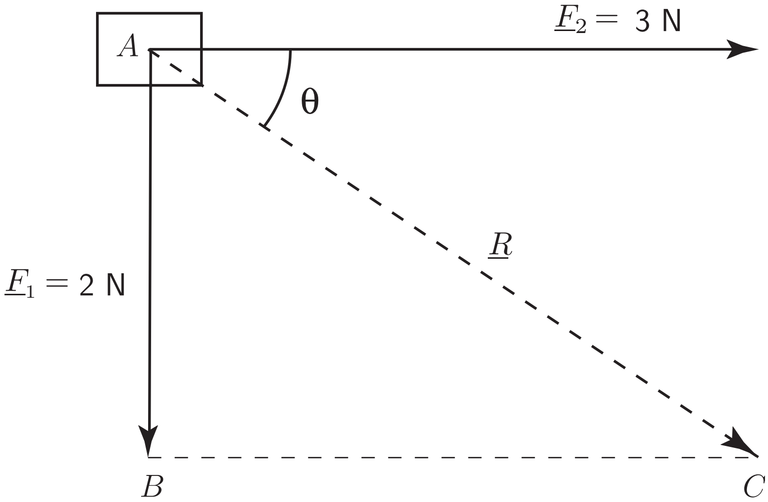


Figure 12

We can use vector addition to find the combined effect or **resultant** of the two concurrent forces. (Concurrent means that the forces act through the same point.) Translating $\underline{F}_2$ until its tail touches the head of $\underline{F}_1$, we complete the triangle ABC as shown. The vector represented by the third side is the resultant, $\underline{R}$. We write

$$\underline{R} = \underline{F_1} + \underline{F_2}$$

and say that $\underline{R}$ is the **vector sum** of $\underline{F_2}$ and $\underline{F_1}$. The resultant force acts at an angle of θ below the horizontal where $\tan\theta = 2/3$, so that $\theta = 33.7°$, and has magnitude (given by Pythagoras' theorem) $\sqrt{13}$ N.

Example: Resolving a force into two perpendicular directions

In the previous Example we saw that two forces acting upon a body can be replaced by a single force which has the same effect. It is sometimes useful to reverse this process and consider a single force as equivalent to two forces acting at right-angles to each other.

Consider the force $\underline{F}$ inclined at an angle θ to the horizontal as shown in Figure 13.

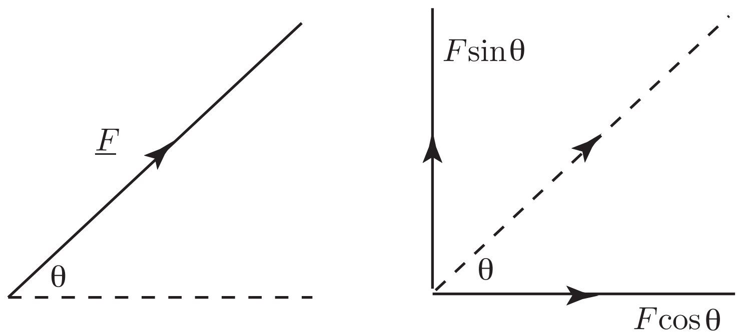


Figure 13

$\underline{F}$ can be replaced by two forces, one of magnitude $F\cos\theta$ and one of magnitude $F\sin\theta$ as shown. We say that $\underline{F}$ has been **resolved into two perpendicular components**. This is sensible because if we re-combine the two perpendicular forces of magnitudes $F\cos\theta$ and $F\sin\theta$ using the triangle law we find $\underline{F}$ to be the resultant force.

For example, Figure 14 shows a force of 5 N acting at an angle of $30°$ to the x axis. It can be resolved into two components, one directed along the x axis with magnitude $5\cos 30°$ and one perpendicular to this of magnitude $5\sin 30°$. Together, these two components have the same effect as the original force.

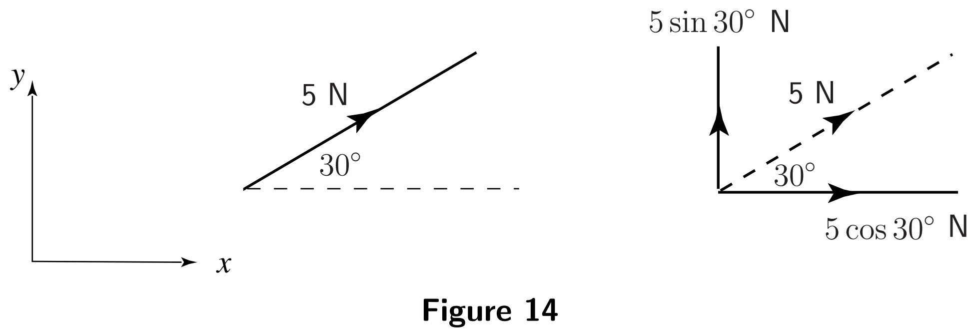
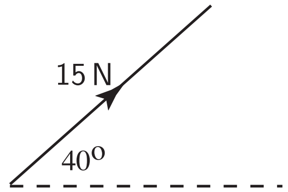


Figure 14

Task

Consider the force shown in the diagram below.



Resolve this force into two perpendicular components, one horizontally to the right, and one vertically upwards.

Your solution

Answer

Horizontal component is $15\cos 40^\circ$ N $= 11.50$ N; vertical component is $15\sin 40^\circ$ N $= 9.64$ N

The need to **resolve** a vector along a given direction occurs in other areas. For example, as a police car or ambulance with siren operating passes by the pitch of the siren appears to increase as the vehicle approaches and decrease as it goes away. This change in pitch is known as the **Doppler effect** This effect occurs in any situation where waves are reflected from a moving object.

A radar gun produces a signal which is bounced off the target moving vehicle so that when it returns to the gun, which also acts as a receiver, it has changed pitch. The speed of the vehicle can be calculated from the change in pitch. The speed indicated on the radar gun is the speed directly towards or away from the gun. However it is not usual to place oneself directly in front of moving vehicle when using the radar gun (Figure 15(a).) Consequently the gun is used at an angle to the line of traffic (Figure 15(b).)

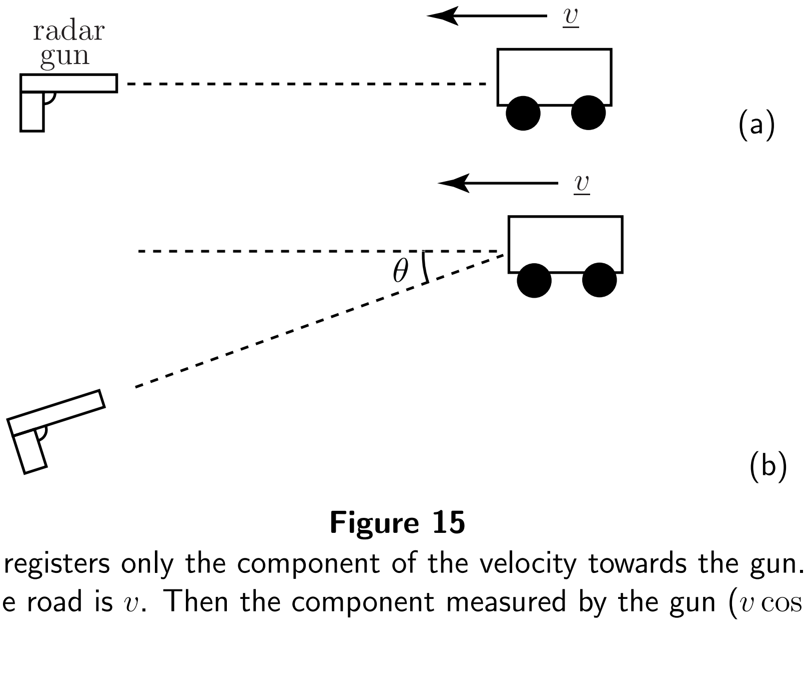


Figure 15

This means that it registers only the component of the velocity towards the gun. Suppose that the true speed along the road is v. Then the component measured by the gun ($v\cos\theta$) is less than v.

Example 2

A safety inspector wishes to check the speed of a train along a straight piece of track. She stands 10 m to the side of the track and uses a radar gun. If the reading on the gun is to be within 5% of the true speed of the train, how far away from the approaching train should the reading be taken?

Solution

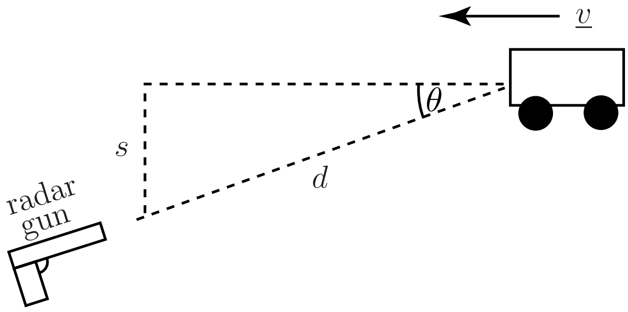


Figure 16

For an error of 5%, the gun should read $0.95|\underline{v}|$. So

$$|\underline{v}|\cos\theta = 0.95|\underline{v}| \quad \text{or} \quad \cos\theta = 0.95$$

If the distance to the side of the track at which the gun is used is s m and the distance between the radar gun and the train is d m, then

$$\sin\theta = \frac{s}{d}$$

Here $s = 10$ so

$$\sqrt{1-\cos^2\theta} = \frac{10}{d}.$$

So, if $\cos\theta = 0.95$

$$d = \frac{10}{\sqrt{1-(0.95)^2}} = 32.03$$

This means that the reading should be taken when the train is over 32 m from the radar gun to ensure an error of less than 5%.

The force vectors on an aeroplane in steady flight

The forces acting on an aeroplane are shown in Figure 17.

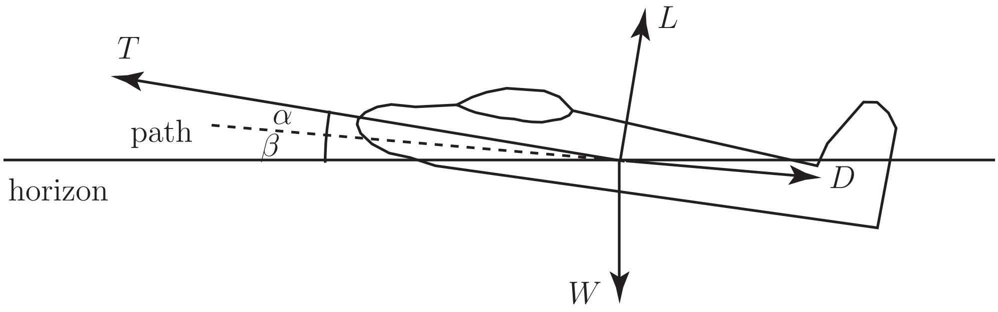


Figure 17

The **magnitude** (strength) of the forces are indicated by

T: the thrust provided by the engines,

W: the weight,

D: the drag (acting against the direction of flight) and

L: the lift (taken perpendicular to the path.)

In a more realistic situation force vectors in *three* dimensions would need to be considered. These are introduced later in this Workbook.

As the plane is in *steady* flight the sum of the forces in any direction is zero. (If this were not the case, then, by Newton's second law, the non-zero resultant force would cause the aeroplane to accelerate.)

So, resolving forces in the direction of the path:

$$T\cos\alpha - D - W\sin\beta = 0$$

Then, resolving forces perpendicular to the path:

$$T\sin\alpha + L - W\cos\beta = 0$$

If the plane has mass 72 000 tonnes, the drag is 130 kN, the lift is 690 kN and $\beta = 6^o$ find the magnitude of the thrust and the value of α to maintain steady flight. From these two equations we see:

$$T\cos\alpha = D + W\sin\beta = 130000 + (72000)(9.81)\sin 6^o = 203830.54$$

and

$$T\sin\alpha = W\cos\beta - L = (72000)(9.81)\cos 6^o - 690000 = 12450.71$$

hence

$$\tan\alpha = \frac{T\sin\alpha}{T\cos\alpha} = \frac{12450.71}{203830.54} = 0.061084 \quad \rightarrow \quad \alpha = 3.50^0$$

and consequently, for the thrust:

$T = 204210$ N.

4. Subtraction of vectors

Subtraction of one vector from another is performed by adding the corresponding negative vector. That is, if we seek $\underline{a} - \underline{b}$ we form $\underline{a} + (-\underline{b})$. This is shown geometrically in Figure 18. Note that in the right-hand diagram the arrow on $\underline{b}$ has been reversed to give $-\underline{b}$.

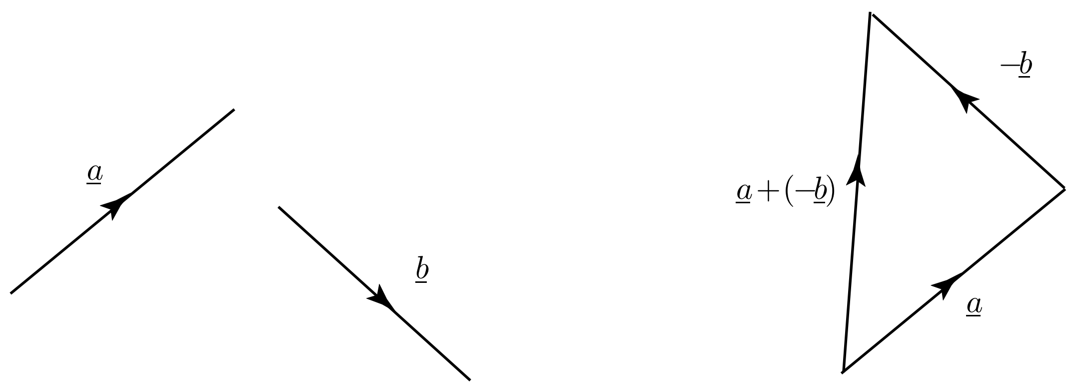


Figure 18: Subtraction of a vector is performed by adding a negative vector

Exercises

1. Vectors $\underline{p}$ and $\underline{q}$ represent two perpendicular sides of a square. Find vector expressions which represent the diagonals of the square.

2. In the rectangle $ABCD$, side AB is represented by the vector $\underline{p}$ and side BC is represented by the vector $\underline{q}$. State the physical significance of the vectors $\underline{p} - \underline{q}$ and $\underline{p} + \underline{q}$.

3. An object is positioned at the origin of a set of axes. Two forces act upon it. The first has magnitude 9 N and acts in the direction of the positive y axis. The second has magnitude 4 N and acts in the direction of the negative x axis. Calculate the magnitude and direction of the resultant force.

4. An object moves in the xy plane with a velocity of 15 m s^{-1} in a direction 48° above the positive x axis. Resolve this velocity into two components, one along the x axis and one along the y axis.

Answers

1. $\underline{p} + \underline{q}$, $\underline{q} - \underline{p}$. Acceptable answers are also $-(\underline{p} + \underline{q})$, $\underline{p} - \underline{q}$.

2. $\underline{p} + \underline{q}$ is the diagonal AC, $\underline{p} - \underline{q}$ is the diagonal DB.

3. Magnitude $\sqrt{97}$, at an angle 66° above the negative x axis.

4. 10.04 m s^{-1} along the x axis, and 11.15 m s^{-1} along the y axis.

5. Multiplying a vector by a scalar

If k is any positive scalar and $\underline{a}$ is a vector then $k\underline{a}$ is a vector in the same direction as $\underline{a}$ but k times as long. If k is negative, $k\underline{a}$ is a vector in the opposite direction to $\underline{a}$ and k times as long. See Figure 19. The vector $k\underline{a}$ is said to be a **scalar multiple** of $\underline{a}$.

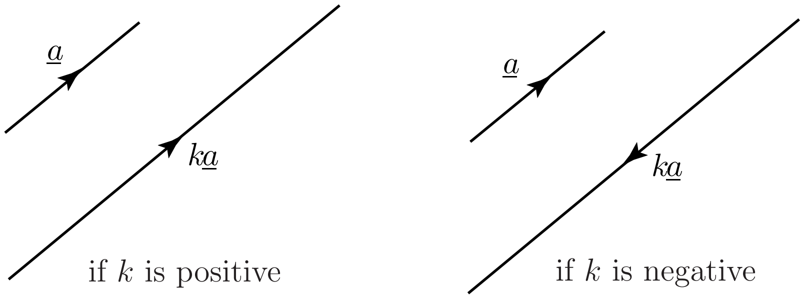


Figure 19: Multiplying a vector by a scalar

The vector $3\underline{a}$ is three times as long as $\underline{a}$ and has the same direction. The vector $\frac{1}{2}\underline{r}$ is in the same direction as $\underline{r}$ but is half as long. The vector $-4\underline{b}$ is in the opposite direction to $\underline{b}$ and four times as long.

For any scalars k and l, and any vectors $\underline{a}$ and $\underline{b}$, the following rules hold:

Key Point 2

$$k(\underline{a}+\underline{b}) = k\underline{a}+k\underline{b}$$

$$(k+l)\underline{a} = k\underline{a}+l\underline{a}$$

$$k(l)\underline{a} = (kl)\underline{a}$$

Task

Using the rules given in Key Point 2, simplify the following:

(a) $3\underline{a}+7\underline{a}$ (b) $2(7\underline{b})$ (c) $4\underline{q}+4\underline{r}$

Your solution

Answer

(a) Using the second rule, $3\underline{a}+7\underline{a}$ can be simplified to $(3+7)\underline{a} = 10\underline{a}$.

(b) Using the third rule $2(7\underline{b}) = (2\times 7)\underline{b} = 14\underline{b}$.

(c) Using the first rule $4\underline{q}+4\underline{r} = 4(\underline{q}+\underline{r})$.

Unit vectors

A vector which has a magnitude of 1 is called a **unit vector**. If $\underline{a}$ has magnitude 3, then a unit vector in the direction of $\underline{a}$ is $\frac{1}{3}\underline{a}$, as shown in Figure 20.

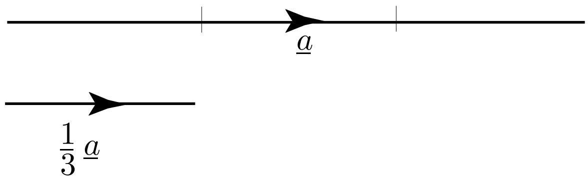


Figure 20: A unit vector has length one unit

A unit vector in the direction of a given vector is found by dividing the given vector by its magnitude: A unit vector in the direction of $\underline{a}$ is given the 'hat' symbol $\hat{\underline{a}}$.

Key Point 3

A unit vector can be found by dividing a vector by its modulus.

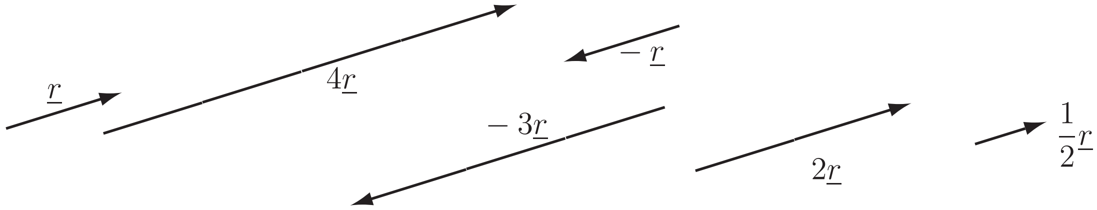
$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

Exercises

1. Draw an arbitrary vector $\underline{r}$. On your diagram draw $2\underline{r}$, $4\underline{r}$, $-\underline{r}$, $-3\underline{r}$ and $\frac{1}{2}\underline{r}$.
2. In triangle OAB the point P divides AB internally in the ratio $m : n$. If $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$ depict this on a diagram and then find an expression for $\overrightarrow{OP}$ in terms of $\underline{a}$ and $\underline{b}$.

Answers

1.



2. $\overrightarrow{OP} = \underline{a} + \dfrac{m}{m+n}(\underline{b} - \underline{a}) = \dfrac{n\underline{a} + m\underline{b}}{m+n}$.

Cartesian Components of Vectors

9.2

Introduction

It is useful to be able to describe vectors with reference to specific coordinate systems, such as the Cartesian coordinate system. So, in this Section, we show how this is possible by defining unit vectors in the directions of the x and y axes. Any other vector in the xy plane can then be represented as a combination of these **basis vectors**. The idea is then extended to three dimensional vectors. This is useful because most engineering problems involve 3D situations.

Prerequisites

Before starting this Section you should ...

- be able to distinguish between a vector and a scalar
- be able torepresent a vector as a directed line segment
- understand the Cartesian coordinate system

Learning Outcomes

On completion you should be able to ...

- explain the meaning of the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$
- express two dimensional and three dimensional vectors in Cartesian form
- find the modulus of a vector expressed in Cartesian form
- find a 'position vector'

1. Two-dimensional coordinate frames

Figure 21 shows a two-dimensional coordinate frame. Any point P in the xy plane can be defined in terms of its x and y coordinates.

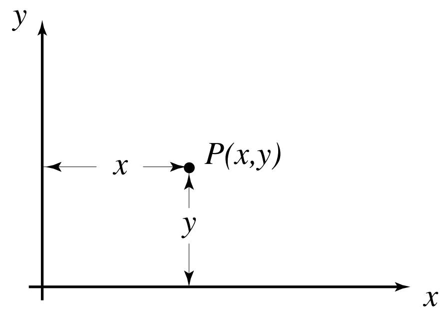


Figure 21

A unit vector pointing in the positive direction of the x-axis is denoted by $\underline{i}$. (Note that it is common practice to write this particular unit vector without the hat $\hat{}$.) It follows that any vector in the direction of the x-axis will be a multiple of $\underline{i}$. Figure 22 shows vectors $\underline{i}$, $2\underline{i}$, $5\underline{i}$ and $-3\underline{i}$. In general a vector of length ℓ in the direction of the x-axis can be written $\ell\underline{i}$.

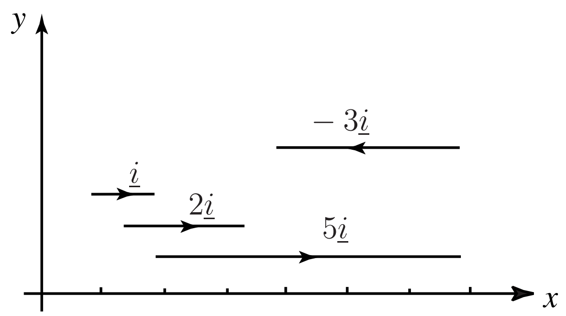


Figure 22: All these vectors are multiples of $\underline{i}$

Similarly, a unit vector pointing in the positive y-axis is denoted by $\underline{j}$. So any vector in the direction of the y-axis will be a multiple of $\underline{j}$. Figure 23 shows $\underline{j}$, $4\underline{j}$ and $-2\underline{j}$. In general a vector of length ℓ in the direction of the y-axis can be written $\ell\underline{j}$.

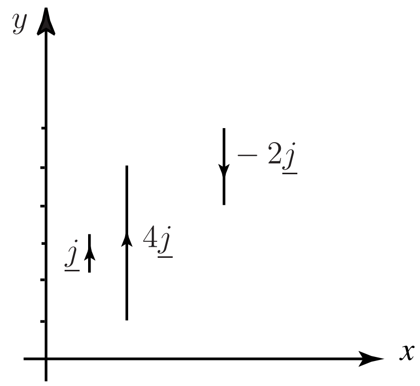


Figure 23: All these vectors are multiples of $\underline{j}$

Key Point 4

$\underset{\sim}{i}$ represents a unit vector in the direction of the positive x-axis
$\underset{\sim}{j}$ represents a unit vector in the direction of the positive y-axis

Example 3

Draw the vectors $5\underset{\sim}{i}$ and $4\underset{\sim}{j}$. Use your diagram and the triangle law of addition to add these two vectors together. First draw the vectors $5\underset{\sim}{i}$ and $4\underset{\sim}{j}$. Then, by translating the vectors so that they lie head to tail, find the vector sum $5\underset{\sim}{i} + 4\underset{\sim}{j}$.

Solution

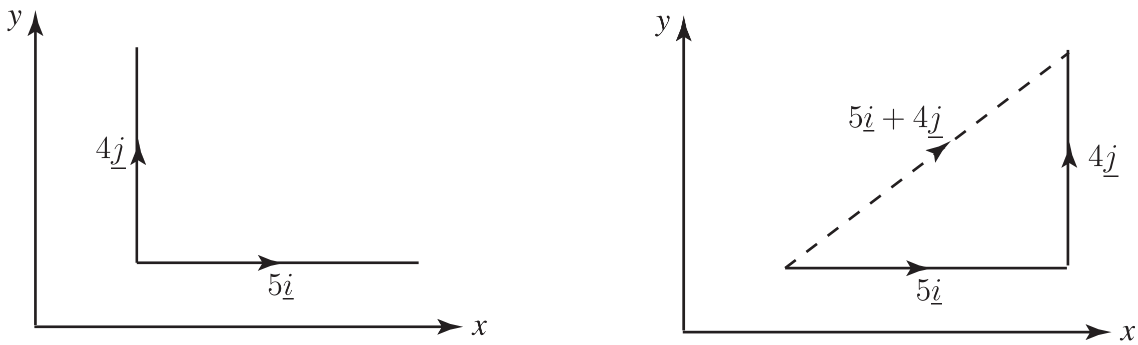


Figure 24

We now generalise the situation in Example 3. Consider Figure 25.
It shows a vector $\underset{\sim}{r} = \overrightarrow{AB}$. We can regard $\underset{\sim}{r}$ as being the resultant of the two vectors $\overrightarrow{AC} = a\underset{\sim}{i}$, and $\overrightarrow{CB} = b\underset{\sim}{j}$. From the triangle law of vector addition

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AC} + \overrightarrow{CB} \\ &= a\underset{\sim}{i} + b\underset{\sim}{j}\end{aligned}$$

We conclude that any vector in the xy plane can be expressed in the form $\underset{\sim}{r} = a\underset{\sim}{i} + b\underset{\sim}{j}$. The numbers a and b are called the **components** of $\underset{\sim}{r}$ in the x and y directions. Sometimes, for emphasis, we will use a_x and a_y instead of a and b to denote the components in the x- and y-directions respectively. In that case we would write $\underset{\sim}{r} = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$.

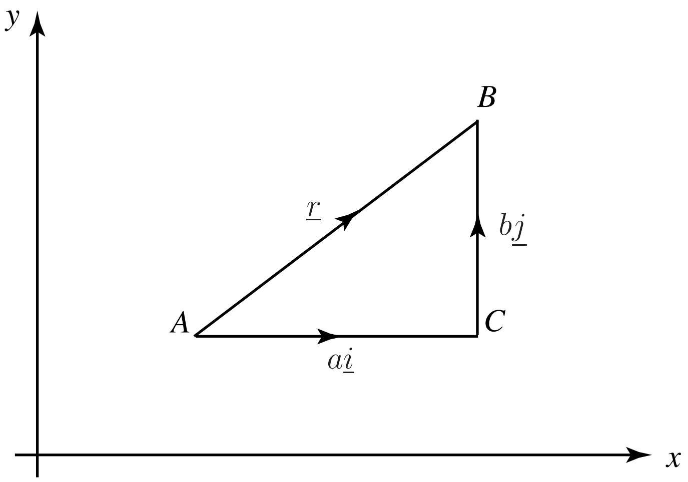


Figure 25: $\underline{AB} = \underline{AC} + \underline{CB}$ by the triangle law

Column vector notation

An alternative, useful, and often briefer notation is to write the vector $\underline{r} = a\underline{i} + b\underline{j}$ in **column vector** notation as

$$\underline{r} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Task

(a) Draw an xy plane and show the vectors $\underline{p} = 2\underline{i} + 3\underline{j}$, and $\underline{q} = 5\underline{i} + \underline{j}$.
(b) Express $\underline{p}$ and $\underline{q}$ using column vector notation.
(c) By translating one of the vectors apply the triangle law to show the sum $\underline{p} + \underline{q}$.
(d) Express the resultant $\underline{p} + \underline{q}$ in terms of $\underline{i}$ and $\underline{j}$.

(a) Draw the xy plane and the required vectors. (They can be drawn from any point in the plane):

Your solution

(b) The column vector form of $\underline{p}$ is $\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$. Write down the column vector form of $\underline{q}$:

Your solution

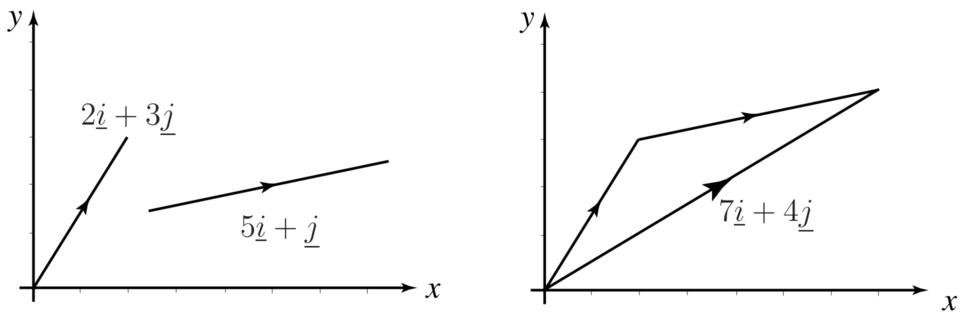
Answer

$\underline{p} = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right) \quad \underline{q} = \left(\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right)$

(c) Translate one of the vectors in part (a) so that they lie head to tail, completing the third side of the triangle to give the resultant $\underline{p}+\underline{q}$:

Your solution

Answer
Note that the vectors have not been drawn to scale.



(d) By studying your diagram note that the resultant has two components $7\underline{i}$, horizontally, and $4\underline{j}$ vertically. Hence write down an expression for $\underline{p}+\underline{q}$:

Your solution

Answer
$7\underline{i}+4\underline{j}$

It is very important to note from the last task that vectors in Cartesian form can be added by simply adding their respective $\underline{i}$ and $\underline{j}$ components.
Thus, if $\underline{a}=a_x\underline{i}+a_y\underline{j}$ and $\underline{b}=b_x\underline{i}+b_y\underline{j}$ then

$$\underline{a}+\underline{b}=(a_x+b_x)\underline{i}+(a_y+b_y)\underline{j}$$

A similar, and obvious, rule applies when subtracting:

$$\underline{a}-\underline{b}=(a_x-b_x)\underline{i}+(a_y-b_y)\underline{j}.$$

Task

If $\underline{a} = 9\underline{i} + 7\underline{j}$ and $\underline{b} = 8\underline{i} + 3\underline{j}$ find (a) $\underline{a} + \underline{b}$ (b) $\underline{a} - \underline{b}$

Your solution

Answer

(a) Simply add the respective components: $17\underline{i} + 10\underline{j}$,

(b) Simply subtract the respective components: $\underline{i} + 4\underline{j}$

Now consider the special case when $\underline{r}$ represents the vector from the origin O to the point $P(a, b)$. This vector is known as the **position vector** of P and is shown in Figure 26.

Key Point 5

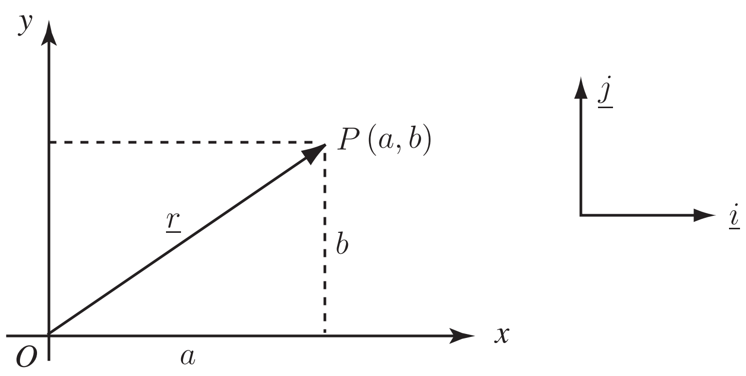


Figure 26

The position vector of P with coordinates (a, b) is $\underline{r} = \overrightarrow{OP} = a\underline{i} + b\underline{j}$

Unlike most vectors, position vectors cannot be freely translated. Because they indicate the position of a point they are **fixed** vectors in the sense that the tail of a position vector is always located at the origin.

Example 4

State the position vectors of the points with coordinates

(a) $P(2,4)$, (b) $Q(-1,5)$, (c) $R(-1,-7)$, (d) $S(8,-4)$.

Solution

(a) $2\underline{i}+4\underline{j}$. (b) $-\underline{i}+5\underline{j}$. (c) $-\underline{i}-7\underline{j}$. (d) $8\underline{i}-4\underline{j}$.

Example 5

Sketch the position vectors $\underline{r}_1 = 3\underline{i}+4\underline{j}$, $\underline{r}_2 = -2\underline{i}+5\underline{j}$, $\underline{r}_3 = -3\underline{i}-2\underline{j}$.

Solution

The vectors are shown below. Note that all position vectors start at the origin.

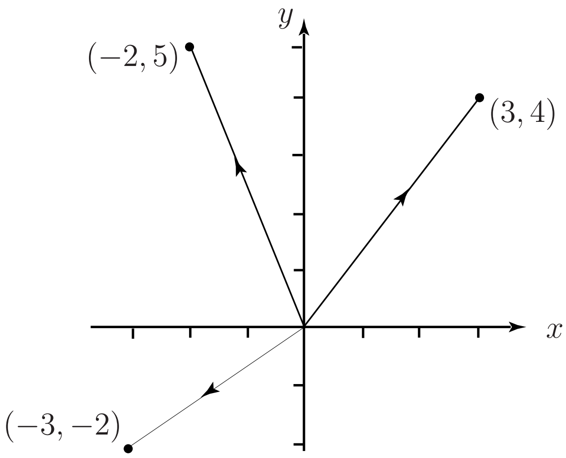


Figure 27

The **modulus** of any vector $\underline{r}$ is equal to its length. As we have noted earlier, the modulus of $\underline{r}$ is usually denoted by $|\underline{r}|$. When $\underline{r} = a\underline{i}+b\underline{j}$ the modulus can be obtained using Pythagoras' theorem. If $\underline{r}$ is the position vector of point P then the modulus is, clearly, the distance of P from the origin.

Key Point 6

If $\underline{r} = a\underline{i}+b\underline{j}$ then $|\underline{r}| = \sqrt{a^2+b^2}$

Example 6

Find the modulus of each of the vectors shown in Example 5.

Solution

(a) $|\underline{r}_1| = |3\underline{i} + 4\underline{j}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$ (b) $|\underline{r}_2| = \sqrt{(-2)^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}.$

(c) $|\underline{r}_3| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$

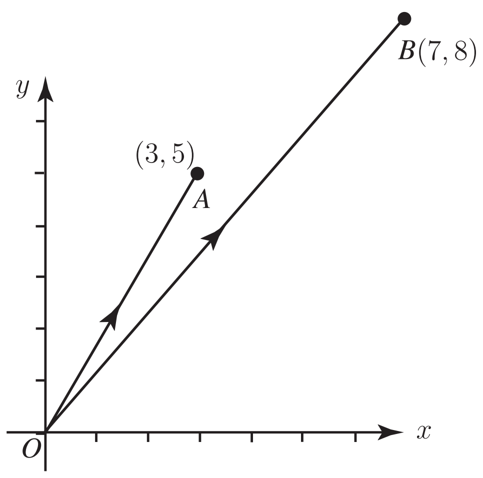
Task

Point A has coordinates $(3, 5)$. Point B has coordinates $(7, 8)$.

(a) Draw a diagram which shows points A and B and draw the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$:

Your solution

Answer



(b) State the position vectors of A and B:

Your solution

$\overrightarrow{OA} =$

$\overrightarrow{OB} =$

Answer

$\overrightarrow{OA} = \underline{a} = 3\underline{i} + 5\underline{j}, \quad \overrightarrow{OB} = \underline{b} = 7\underline{i} + 8\underline{j}$

(c) Referring to your figure and using the triangle law you can write $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ so that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. Hence write down an expression for $\overrightarrow{AB}$ in terms of the unit vectors $\underline{i}$ and $\underline{j}$:

Your solution

Answer

$\overrightarrow{AB} = (7\underline{i} + 8\underline{j}) - (3\underline{i} + 5\underline{j}) = 4\underline{i} + 3\underline{j}$

(d) Calculate the length of $\overrightarrow{AB} = |4\underline{i} + 3\underline{j}|$:

Your solution

Answer

$|\overrightarrow{AB}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

Exercises

1. Explain the distinction between a position vector, and a more general free vector.

2. What is meant by the symbols $\underline{i}$ and $\underline{j}$?

3. State the position vectors of the points with coordinates

   (a) $P(4,7)$, (b) $Q(-3,5)$, (c) $R(0,3)$, (d) $S(-1,0)$

4. State the coordinates of the point P if its position vector is:

   (a) $3\underline{i} - 7\underline{j}$, (b) $-4\underline{i}$, (c) $-0.5\underline{i} + 13\underline{j}$, (d) $a\underline{i} + b\underline{j}$

5. Find the modulus of each of the following vectors:

   (a) $\underline{r} = 7\underline{i} + 3\underline{j}$, (b) $\underline{r} = 17\underline{i}$, (c) $\underline{r} = 2\underline{i} - 3\underline{j}$, (d) $\underline{r} = -3\underline{j}$,
   (e) $\underline{r} = a\underline{i} + b\underline{j}$, (f) $\underline{r} = a\underline{i} - b\underline{j}$

6. Point P has coordinates $(7,8)$. Point Q has coordinates $(-2,4)$.

   (a) Draw a sketch showing vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$

   (b) State the position vectors of P and Q,

   (c) Find an expression for $\overrightarrow{PQ}$,

   (d) Find $|\overrightarrow{PQ}|$.

Answers

1. Free vectors may be translated provided their direction and length remain unchanged. Position vectors must always start at the origin.

2. $\underline{i}$ is a unit vector in the direction of the positive x-axis. $\underline{j}$ is a unit vector in the direction of the positive y-axis.

3. (a) $4\underline{i} + 7\underline{j}$, (b) $-3\underline{i} + 5\underline{j}$, (c) $3\underline{j}$, (d) $-\underline{i}$.

4. (a) $(3,-7)$, (b) $(-4,0)$, (c) $(-0.5,13)$, (d) (a,b)

5. (a) $\sqrt{58}$, (b) 17, (c) $\sqrt{13}$, (d) 3, (e) $\sqrt{a^2+b^2}$, (f) $\sqrt{a^2+b^2}$.

6. (b) $\overrightarrow{OP} = 7\underline{i} + 8\underline{j}$ and $\overrightarrow{OQ} = -2\underline{i} + 4\underline{j}$, (c) $\overrightarrow{PQ} = -9\underline{i} - 4\underline{j}$, (d) $|\overrightarrow{PQ}| = \sqrt{97}$.

2. Three-dimensional coordinate frames

The real world is three-dimensional and in order to solve many engineering problems it is necessary to develop expertise in the mathematics of three-dimensional space. An important application of vectors is their use to locate points in three dimensions. When two distinct points are known we can draw a unique straight line between them. Three distinct points which do not lie on the same line form a unique plane. Vectors can be used to describe points, lines, and planes in three dimensions. These mathematical foundations underpin much of the technology associated with computer graphics and the control of robots. In this Section we shall introduce the vector methods which underlie these applications.

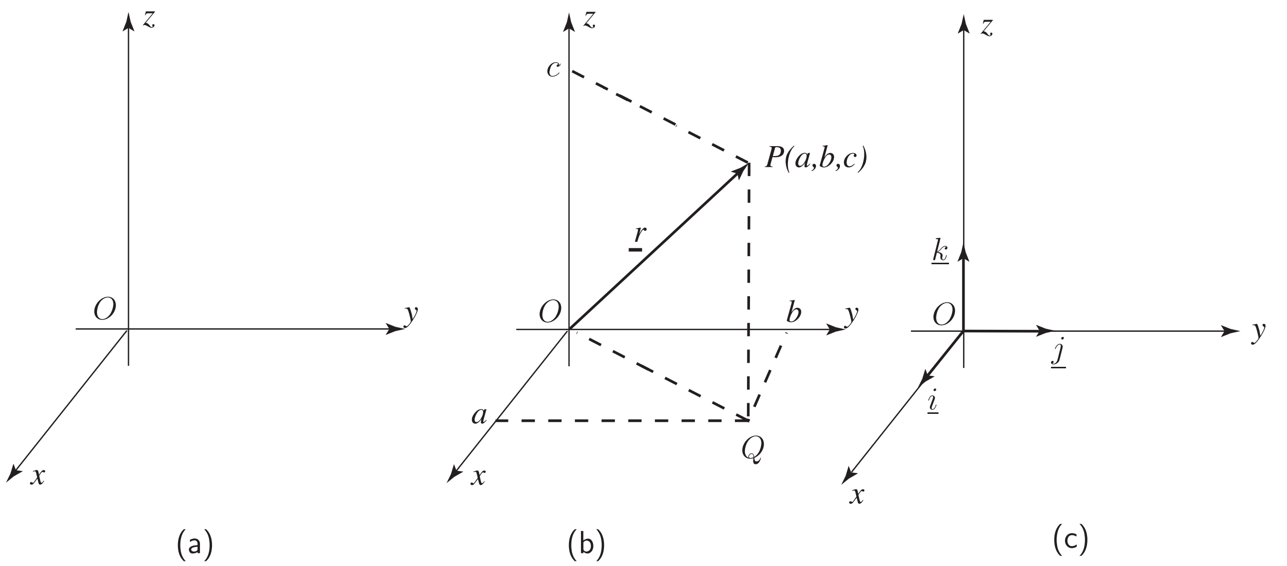


Figure 28

Figure 28(a) shows a three-dimensional coordinate frame. Note that the third dimension requires the addition of a third axis, the z-axis. Although these three axes are drawn in the plane of the paper you should remember that we are now thinking of three-dimensional situations. Just as in two-dimensions the x and y axes are perpendicular, in three dimensions the x,y and z axes are all perpendicular to each other. We say they are **mutually perpendicular.** There is no reason why we could not have chosen the z-axis in the opposite direction. However, it is conventional to choose the directions shown in Figure 28(a).

Any point in the three dimensional space can be defined in terms of its x, y and z coordinates. Consider the point P with coordinates (a, b, c) as shown in Figure 28(b). The vector from the origin to the point P is known as the position vector of P, denoted $\overrightarrow{OP}$ or $\underline{r}$. To arrive at P from O we can think of moving a units in the x direction, b units in the y direction and c units in the z direction. A **unit vector** pointing in the positive direction of the z-axis is denoted by $\underline{k}$. See the Figure 28(c). Noting that $\overrightarrow{OQ} = a\underline{i} + b\underline{j}$ and that $\overrightarrow{QP} = c\underline{k}$ we can state

$$\begin{aligned}\underline{r} = \overrightarrow{OP} &= \overrightarrow{OQ} + \overrightarrow{QP} \\ &= a\underline{i} + b\underline{j} + c\underline{k}\end{aligned}$$

We conclude that the position vector of the point with coordinates (a, b, c) is $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$. (We might, for convenience, sometimes use a subscript notation. For example we might refer to the position vector $\underline{r}$ as $\underline{r} = r_x\underline{i} + r_y\underline{j} + r_z\underline{k}$ in which (r_x, r_y, r_z) have taken the place of (a, b, c).)

Key Point 7

If P has coordinates (a, b, c) then its position vector is

$$\underline{r} = \overrightarrow{OP} = a\underline{i} + b\underline{j} + c\underline{k}$$

Task

State the position vector of the point with coordinates $(9, -8, 6)$.

Your solution

Answer

$9\underline{i} - 8\underline{j} + 6\underline{k}$

The modulus of the vector $\overrightarrow{OP}$ is equal to the distance OP, which can be obtained by Pythagoras' theorem:

Key Point 8

If $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ then

$$|\underline{r}| = \sqrt{a^2 + b^2 + c^2}$$

Task

Find the modulus of the vector $\underline{r} = 4\underline{i} + 2\underline{j} + 3\underline{k}$.

Your solution

Answer

$|\underline{r}| = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{16 + 4 + 9} = \sqrt{29}$

Example 7

Points A, B and C have coordinates $(-1,1,4)$, $(8,0,2)$ and $(5,-2,11)$ respectively.

(a) Find the position vectors of A, B and C.

(b) Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(c) Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$.

Solution

(a) Denoting the position vectors of A, B and C by $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively, we find

$$\underline{a} = -\underline{i} + \underline{j} + 4\underline{k}, \quad \underline{b} = 8\underline{i} + 2\underline{k}, \quad \underline{c} = 5\underline{i} - 2\underline{j} + 11\underline{k}$$

(b) $\overrightarrow{AB} = \underline{b} - \underline{a} = 9\underline{i} - \underline{j} - 2\underline{k}$. $\overrightarrow{BC} = \underline{c} - \underline{b} = -3\underline{i} - 2\underline{j} + 9\underline{k}$.

(c) $|\overrightarrow{AB}| = \sqrt{9^2 + (-1)^2 + (-2)^2} = \sqrt{86}$. $|\overrightarrow{BC}| = \sqrt{(-3)^2 + (-2)^2 + 9^2} = \sqrt{94}$.

Exercises

1. State the position vector of the point with coordinates $(4,-4,3)$.
2. Find the modulus of each of the following vectors.

   (a) $7\underline{i} + 2\underline{j} + 3\underline{k}$, (b) $7\underline{i} - 2\underline{j} + 3\underline{k}$, (c) $2\underline{j} + 8\underline{k}$, (d) $-\underline{i} - 2\underline{j} + 3\underline{k}$, (e) $a\underline{i} + b\underline{j} + c\underline{k}$
3. Points P, Q and R have coordinates $(9,1,0)$, $(8,-3,5)$, and (5,5,7) respectively.

   (a) Find the position vectors $\underline{p}$, $\underline{q}$, $\underline{r}$ of P, Q and R,

   (b) Find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$

   (c) Find $|\overrightarrow{PQ}|$ and $|\overrightarrow{QR}|$.

Answers

1. $4\underline{i} - 4\underline{j} + 3\underline{k}$
2. (a) $\sqrt{62}$, (b) $\sqrt{62}$, (c) $\sqrt{68}$, (d) $\sqrt{14}$, (e) $\sqrt{a^2 + b^2 + c^2}$.
3. (a) $\underline{p} = 9\underline{i} + \underline{j}$, $\underline{q} = 8\underline{i} - 3\underline{j} + 5\underline{k}$, $\underline{r} = 5\underline{i} + 5\underline{j} + 7\underline{k}$.

   (b) $\overrightarrow{PQ} = -\underline{i} - 4\underline{j} + 5\underline{k}$, $\overrightarrow{QR} = -3\underline{i} + 8\underline{j} + 2\underline{k}$

   (c) $|\overrightarrow{PQ}| = \sqrt{42}$, $|\overrightarrow{QR}| = \sqrt{77}$

The Scalar Product

9.3

Introduction

There are two kinds of multiplication involving vectors. The first is known as the **scalar product** or **dot product**. This is so-called because when the scalar product of two vectors is calculated the result is a scalar. The second product is known as the **vector product**. When this is calculated the result is a vector. The definitions of these products may seem rather strange at first, but they are widely used in applications. In this Section we consider only the scalar product.

Prerequisites

Before starting this Section you should ...

- know that a vector can be represented as a directed line segment
- know how to express a vector in Cartesian form
- know how to find the modulus of a vector

Learning Outcomes

On completion you should be able to ...

- calculate, from its definition, the scalar product of two given vectors
- calculate the scalar product of two vectors given in Cartesian form
- use the scalar product to find the angle between two vectors
- use the scalar product to test whether two vectors are perpendicular

1. Definition of the scalar product

Consider the two vectors $\underline{a}$ and $\underline{b}$ shown in Figure 29.

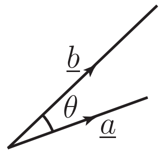


Figure 29: Two vectors subtend an angle θ

Note that the tails of the two vectors coincide and that the angle between the vectors is labelled θ. Their scalar product, denoted by $\underline{a} \cdot \underline{b}$, is defined as the product $|\underline{a}|\,|\underline{b}|\,\cos\theta$. It is very important to use the **dot** in the formula. The dot is the specific symbol for the scalar product, and is the reason why the scalar product is also known as the **dot product**. You should not use a $\times$ sign in this context because this sign is reserved for the vector product which is quite different.

The angle θ is always chosen to lie between 0 and π, and the tails of the two vectors must coincide. Figure 30 shows two incorrect ways of measuring θ.

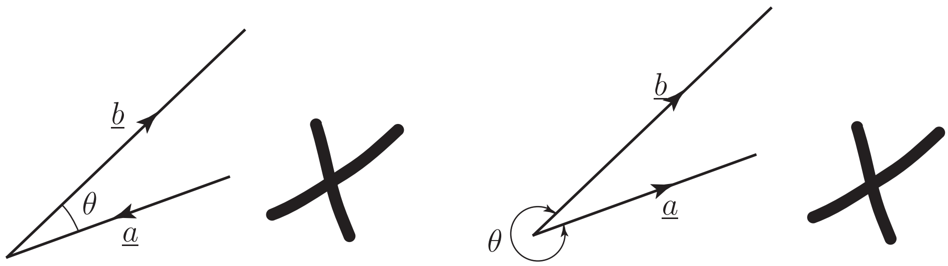


Figure 30: θ should not be measured in these ways

Key Point 9

The scalar product of $\underline{a}$ and $\underline{b}$ is: $\quad \underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta$

We can remember this formula as:

> "The modulus of the first vector, multiplied by the modulus of the second vector, multiplied by the cosine of the angle between them."

Clearly $\underline{b} \cdot \underline{a} = |\underline{b}|\,|\underline{a}|\,\cos\theta$ and so

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}.$$

Thus we can evaluate a scalar product in any order: the operation is **commutative**.

Example 8

Vectors $\underline{a}$ and $\underline{b}$ are shown in the Figure 31. Vector $\underline{a}$ has modulus 6 and vector $\underline{b}$ has modulus 7 and the angle between them is $60°$. Calculate $\underline{a}.\underline{b}$.

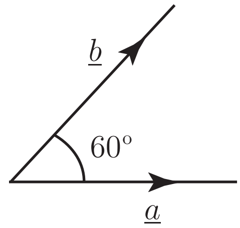


Figure 31

Solution

The angle between the two vectors is $60°$. Hence

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta = (6)(7)\cos 60° = 21$$

The scalar product of $\underline{a}$ and $\underline{b}$ is 21. Note that a scalar product is always a scalar.

Example 9

Find $\underline{i} \cdot \underline{i}$ where $\underline{i}$ is the unit vector in the direction of the positive x axis.

Solution

Because $\underline{i}$ is a unit vector its modulus is 1. Also, the angle between $\underline{i}$ and itself is zero. Therefore

$$\underline{i}.\underline{i} = (1)(1)\cos 0° = 1$$

So the scalar product of $\underline{i}$ with itself equals 1. It is easy to verify that $\underline{j}.\underline{j} = 1$ and $\underline{k}.\underline{k} = 1$.

Example 10

Find $\underline{i} \cdot \underline{j}$ where $\underline{i}$ and $\underline{j}$ are unit vectors in the directions of the x and y axes.

Solution

Because $\underline{i}$ and $\underline{j}$ are unit vectors they each have a modulus of 1. The angle between the two vectors is $90°$. Therefore

$$\underline{i} \cdot \underline{j} = (1)(1)\cos 90° = 0$$

That is $\underline{i} \cdot \underline{j} = 0$.

The following results are easily verified:

Key Point 10

$$\begin{aligned}\underline{i}\cdot\underline{i} &= \underline{j}\cdot\underline{j}=\underline{k}\cdot\underline{k}=1\\ \underline{i}\cdot\underline{j} &= \underline{j}\cdot\underline{i}=0\\ \underline{i}\cdot\underline{k} &= \underline{k}\cdot\underline{i}=0\\ \underline{j}\cdot\underline{k} &= \underline{k}\cdot\underline{j}=0\end{aligned}$$

Generally, whenever any two vectors are perpendicular to each other their scalar product is zero because the angle between the vectors is 90° and $\cos 90^\circ = 0$.

Key Point 11

The scalar product of perpendicular vectors is zero.

2. A formula for finding the scalar product

We can use the results summarized in Key Point 10 to obtain a formula for finding a scalar product when the vectors are given in Cartesian form. We consider vectors in the xy plane. Suppose $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$. Then

$$\begin{aligned}\underline{a}\cdot\underline{b} &= (a_1\underline{i}+a_2\underline{j})\cdot(b_1\underline{i}+b_2\underline{j})\\ &= a_1\underline{i}\cdot(b_1\underline{i}+b_2\underline{j})+a_2\underline{j}\cdot(b_1\underline{i}+b_2\underline{j})\\ &= a_1b_1\underline{i}\cdot\underline{i}+a_1b_2\underline{i}\cdot\underline{j}+a_2b_1\underline{j}\cdot\underline{i}+a_2b_2\underline{j}\cdot\underline{j}\end{aligned}$$

Using the results in Key Point 10 we can simplify this to give the following formula:

Key Point 12

If $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$ then

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2$$

Thus to find the scalar product of two vectors their $\underline{i}$ components are multiplied together, their $\underline{j}$ components are multiplied together and the results are added.

Example 11

If $\underline{a} = 7\underline{i} + 8\underline{j}$ and $\underline{b} = 5\underline{i} - 2\underline{j}$, find the scalar product $\underline{a} \cdot \underline{b}$.

Solution

We use Key Point 12:

$$\underline{a} \cdot \underline{b} = (7\underline{i} + 8\underline{j}) \cdot (5\underline{i} - 2\underline{j}) = (7)(5) + (8)(-2) = 35 - 16 = 19$$

The formula readily generalises to vectors in three dimensions as follows:

Key Point 13

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example 12

If $\underline{a} = 5\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{b} = 8\underline{i} - 9\underline{j} + 11\underline{k}$, find $\underline{a} \cdot \underline{b}$.

Solution

We use the formula in Key Point 13:

$$\underline{a} \cdot \underline{b} = (5)(8) + (3)(-9) + (-2)(11) = 40 - 27 - 22 = -9$$

Note again that the result is a scalar: there are no $\underline{i}$'s, $\underline{j}$'s, or $\underline{k}$'s in the answer.

Task

If $\underline{p} = 4\underline{i} - 3\underline{j} + 7\underline{k}$ and $\underline{q} = 6\underline{i} - \underline{j} + 2\underline{k}$, find $\underline{p} \cdot \underline{q}$.

Use Key Point 13:

Your solution

Answer

41

Task

If $\underline{r} = 3\underline{i} + 2\underline{j} + 9\underline{k}$ find $\underline{r} \cdot \underline{r}$. Show that this is the same as $|\underline{r}|^2$.

Your solution

Answer

$\underline{r} \cdot \underline{r} = (3\underline{i} + 2\underline{j} + 9\underline{k}) \cdot (3\underline{i} + 2\underline{j} + 9\underline{k}) = 3\underline{i} \cdot 3\underline{i} + 3\underline{i} \cdot 2\underline{j} + \cdots = 9 + 0 + \cdots = 94.$

$|\underline{r}| = \sqrt{9 + 4 + 81} = \sqrt{94}$, hence $|\underline{r}|^2 = \underline{r} \cdot \underline{r}$.

The above result is generally true:

Key Point 14

For any vector $\underline{r}$, $\quad |\underline{r}|^2 = \underline{r} \cdot \underline{r}$

3. Resolving one vector along another

The scalar product can be used to find the component of a vector in the direction of another vector. Consider Figure 32 which shows two arbitrary vectors $\underline{a}$ and $\underline{n}$. Let $\hat{\underline{n}}$ be a **unit vector** in the direction of $\underline{n}$.

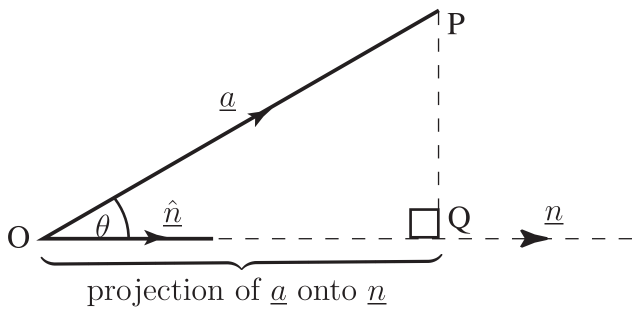


Figure 32

Study the figure carefully and note that a perpendicular has been drawn from P to meet $\underline{n}$ at Q. The distance OQ is called the **projection** of $\underline{a}$ onto $\underline{n}$. Simple trigonometry tells us that the length of the projection is $|\underline{a}|\cos\theta$. Now by taking the scalar product of $\underline{a}$ with the unit vector $\hat{\underline{n}}$ we find

$$\underline{a}\cdot\hat{\underline{n}} = |\underline{a}|\,|\hat{\underline{n}}|\,\cos\theta = |\underline{a}|\cos\theta \qquad \text{(since } |\hat{\underline{n}}| = 1)$$

We conclude that

Key Point 15

Resolving One Vector Along Another

$\underline{a}\cdot\hat{\underline{n}}$ is the component of $\underline{a}$ in the direction of $\underline{n}$

Example 13

Figure 33 shows a plane containing the point A which has position vector $\underline{a}$. The vector $\hat{\underline{n}}$ is a unit vector perpendicular to the plane (such a vector is called a **normal** vector). Find an expression for the perpendicular distance, ℓ, of the plane from the origin.

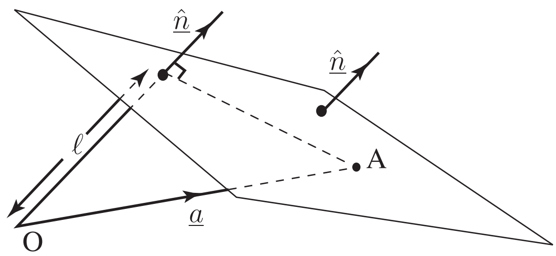


Figure 33

Solution

From the diagram we note that the perpendicular distance ℓ of the plane from the origin is the projection of $\underline{a}$ onto $\hat{\underline{n}}$ and, using Key Point 15, is thus $\underline{a} \cdot \hat{\underline{n}}$.

4. Using the scalar product to find the angle between vectors

We have two distinct ways of calculating the scalar product of two vectors. From Key Point 9 $\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\,\cos\theta$ whilst from Key Point 13 $\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$. Both methods of calculating the scalar product are entirely equivalent and will always give the same value for the scalar product. We can exploit this correspondence to find the angle between two vectors. The following example illustrates the procedure to be followed.

Example 14

Find the angle between the vectors $\underline{a} = 5\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{b} = 8\underline{i} - 9\underline{j} + 11\underline{k}$.

Solution

The scalar product of these two vectors has already been found in Example 12 to be -9. The modulus of $\underline{a}$ is $\sqrt{5^2 + 3^2 + (-2)^2} = \sqrt{38}$. The modulus of $\underline{b}$ is $\sqrt{8^2 + (-9)^2 + 11^2} = \sqrt{266}$. Substituting these values for $\underline{a} \cdot \underline{b}$, $|\underline{a}|$ and $\underline{b}$ into the formula for the scalar product we find

$$\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}|\,|\underline{b}|\,\cos\theta \\ -9 &= \sqrt{38}\,\sqrt{266}\cos\theta \end{aligned}$$

from which

$$\cos\theta = \frac{-9}{\sqrt{38}\sqrt{266}} = -0.0895$$

so that $\quad \theta = \cos^{-1}(-0.0895) = 95.14°$

In general, the angle between two vectors can be found from the following formula:

Key Point 16

The angle θ between vectors $\underline{a}$, $\underline{b}$ is such that:

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|}$$

Exercises

1. If $\underline{a} = 2\underline{i} - 5\underline{j}$ and $\underline{b} = 3\underline{i} + 2\underline{j}$ find $\underline{a} \cdot \underline{b}$ and verify that $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$.
2. Find the angle between $\underline{p} = 3\underline{i} - \underline{j}$ and $\underline{q} = -4\underline{i} + 6\underline{j}$.
3. Use the definition of the scalar product to show that if two vectors are perpendicular, their scalar product is zero.
4. If $\underline{a}$ and $\underline{b}$ are perpendicular, simplify $(\underline{a} - 2\underline{b}) \cdot (3\underline{a} + 5\underline{b})$.
5. If $\underline{p} = \underline{i} + 8\underline{j} + 7\underline{k}$ and $\underline{q} = 3\underline{i} - 2\underline{j} + 5\underline{k}$, find $\underline{p} \cdot \underline{q}$.
6. Show that the vectors $\frac{1}{2}\underline{i} + \underline{j}$ and $2\underline{i} - \underline{j}$ are perpendicular.
7. The work done by a force $\underline{F}$ in moving a body through a displacement $\underline{r}$ is given by $\underline{F} \cdot \underline{r}$. Find the work done by the force $\underline{F} = 3\underline{i} + 7\underline{k}$ if it causes a body to move from the point with coordinates $(1, 1, 2)$ to the point $(7, 3, 5)$.
8. Find the angle between the vectors $\underline{i} - \underline{j} - \underline{k}$ and $2\underline{i} + \underline{j} + 2\underline{k}$.

Answers

1. -4.
2. $142.1°$,
3. This follows from the fact that $\cos\theta = 0$ since $\theta = 90°$.
4. $3a^2 - 10b^2$.
5. 22.
6. This follows from the scalar product being zero.
7. 39 units.
8. $101.1°$

5. Vectors and electrostatics

Electricity is important in several branches of engineering - not only in electrical or electronic engineering. For example the design of the electrostatic precipitator plates for cleaning the solid fuel power stations involves both mechanical engineering (structures and mechanical rapping systems for cleaning the plates) and electrostatics (to determine the electrical forces between solid particles and plates).

The following example and tasks relate to the electrostatic forces between particles. Electric charge is measured in coulombs (C). Charges can be either positive or negative.

The force between two charges

Let q_1 and q_2 be two charges in free space located at points P_1 and P_2. Then q_1 will experience a force due to the presence of q_2 and directed from P_2 towards P_1.
This force is of magnitude $K\,\dfrac{q_1q_2}{r^2}$ where r is the distance between P_1 and P_2 and K is a constant.
In vector notation this coulomb force (measured in newtons) can then be expressed as $\underline{F} = K\dfrac{q_1q_2}{r^2}\,\hat{\underline{r}}$ where $\hat{\underline{r}}$ is a unit vector directed from P_2 towards P_1.
The constant K is known to be $\dfrac{1}{4\pi\varepsilon_0}$ where $\varepsilon_0 = 8.854 \times 10^{-12}$ F m^{-1} (farads per metre).

The electric field

A unit charge located at a general point G will then experience a force $\dfrac{Kq_1}{r_1^2}\hat{\underline{r}}_1$ (where $\hat{\underline{r}}_1$ is the unit vector directed from P_1 towards G) due to a charge q_1 located at P_1. This is the electric field $\underline{E}$ newtons per coulomb (N C^{-1} or alternatively V m^{-1}) at G due to the presence of q_1.
For several point charges q_1 at P_1, q_2 at P_2 etc., the total electric field $\underline{E}$ at G is given by

$$\underline{E} = \frac{Kq_1}{r_1^2}\,\hat{\underline{r}}_1 + \frac{Kq_2}{r_2^2}\,\hat{\underline{r}}_2 + \ldots$$

where $\hat{\underline{r}}_i$ is the unit vector directed from point P_i towards G.
From the definition of a unit vector we see that

$$\underline{E} = \frac{Kq_1}{r_1^2}\,\frac{\underline{r}_1}{|\underline{r}_1|} + \frac{Kq_2}{r_2^2}\,\frac{\underline{r}_2}{|\underline{r}_2|} + \ldots = \frac{Kq_1}{|r_1|^3}\underline{r}_1 + \frac{Kq_2}{|\underline{r}_2|^3}\underline{r}_2 + \ldots = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{|\underline{r}_1|^3}\underline{r}_1 + \frac{q_2}{|\underline{r}_2|^3}\underline{r}_2 + \ldots\right]$$

where $\underline{r}_i$ is the vector directed from point P_i towards G, so that $\underline{r}_1 = \underline{OG} - \underline{OP}_1$ etc., where $\underline{OG}$ and $\underline{OP}_1$ are the position vectors of G and P_1 (see Figure 34).

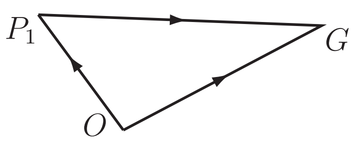


Figure 34

$$\underline{OP_1} + \underline{P_1G} = \underline{OG} \qquad\qquad \underline{P_1G} = \underline{OG} - \underline{OP_1}$$

The work done

The work done W (energy expended) in moving a charge q through a distance dS, in a direction given by the unit vector $\underline{S}/|\underline{S}|$, in an electric field $\underline{E}$ is (defined by)

$$W = -q\underline{E}.\mathrm{d}\underline{S} \qquad (4)$$

where W is in joules.

Engineering Example 1

Field due to point charges

In free space, point charge $q_1 = 10\ n$C ($1\ n$C $= 10^{-9}$C, i.e. a nanocoulomb) is at $P_1(0, -4, 0)$ and charge $q_2 = 20\ n$C is at $P_2 = (0, 0, 4)$.

[Note: Since the x-coordinate of both charges is zero, the problem is two-dimensional in the yz plane as shown in Figure 35.]

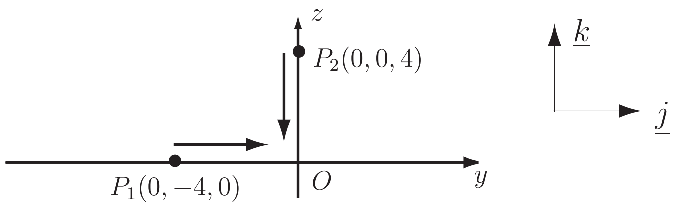


Figure 35

(a) Find the field at the origin $\underline{E}_{1,2}$ due to q_1 and q_2.

(b) Where should a third charge $q_3 = 30\ n$C be placed in the yz plane so that the total field due to q_1, q_2, q_3 is zero at the origin?

Solution

(a) Total field at the origin $\underline{E}_{1,2} =$ (field at origin due to charge at P_1) $+$ (field at origin due to charge at P_2). Therefore

$$\underline{E}_{1,2} = \frac{10 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times 4^2}\underline{j} + \frac{20 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times 4^2}(-\underline{k}) = 5.617\underline{j} - 11.23\underline{k}$$

(The negative sign in front of the second term results from the fact that the direction from P_2 to O is in the $-z$ direction.)

(b) Suppose the third charge $q_3 = 30\ n$C is placed at $P_3(0, a, b)$. The field at the origin due to the third charge is

$$\underline{E}_3 = \frac{30 \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times (a^2 + b^2)} \times \frac{-(a\underline{j} + b\underline{k})}{(a^2 + b^2)^{1/2}},$$

where $\dfrac{a\underline{j} + b\underline{k}}{(a^2 + b^2)^{1/2}}$ is the unit vector in the direction from O to P_3

If the position of the third charge is such that the total field at the origin is zero, then $\underline{E}_3 = -\underline{E}_{1,2}$. There are two unknowns (a and b). We can write down two equations by considering the $\underline{j}$ and $\underline{k}$ directions.

Solution (contd.)

$$\underline{E}_3 = -269.6\left[\frac{a}{(a^2+b^2)^{3/2}}\underline{j} + \frac{b}{(a^2+b^2)^{3/2}}\underline{k}\right] \qquad \underline{E}_{1,2} = 5.617\underline{j} - 11.23\underline{k}$$

So

$$5.617 = 269.6 \times \frac{a}{(a^2+b^2)^{3/2}} \tag{1}$$

$$-11.23 = 269.6 \times \frac{b}{(a^2+b^2)^{3/2}} \tag{2}$$

So

$$\frac{a}{(a^2+b^2)^{3/2}} = 0.02083 \tag{3}$$

$$\frac{b}{(a^2+b^2)^{3/2}} = -0.04165 \tag{4}$$

Squaring and adding (3) and (4) gives $\dfrac{a^2+b^2}{(a^2+b^2)^3} = 0.002169$

So

$$(a^2+b^2) = 21.47 \tag{5}$$

Substituting back from (5) into (1) and (2) gives $a = 2.07$ and $b = -4.14$, to 3 s.f.

Task

Eight point charges of 1 nC each are located at the corners of a cube in free space which is 1 m on each side (see Figure 36). Calculate $|\underline{E}|$ at

(a) the centre of the cube

(b) the centre of any face

(c) the centre of any edge.

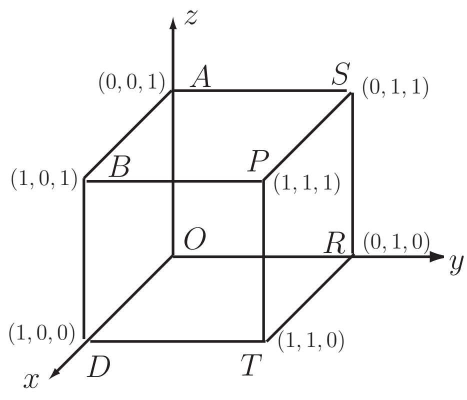


Figure 36

Your solution

Work the problem on a separate piece of paper but record here your main results and conclusions.

Answer

(a) The field at the centre of the cube is zero because of the symmetrical distribution of the charges.

(b) Because of the symmetrical nature of the problem it does not matter which face is chosen in order to find the magnitude of the field at the centre of a face. Suppose the chosen face has corners located at $P(1,1,1),\ T(1,1,0),\ R(0,1,0)$ and $S(0,1,1)$ then the centre (C) of this face can be seen from the diagram to be located at $C\left(\dfrac{1}{2}, 1, \dfrac{1}{2}\right)$.

The electric field at C due to the charges at the corners P, T, R and S will then be zero since the field vectors due to equal charges located at opposite corners of the square $PTRS$ cancel one another out. The field at C is then due to the equal charges located at the remaining four corners $(OABD)$ of the cube, and we note from the symmetry of the cube, that the distance of each of these corners from C will be the same. In particular the distance $OC = \sqrt{\left(\dfrac{1}{2}\right)^2 + 1^2 + \left(\dfrac{1}{2}\right)^2} = \sqrt{1.5}$ m. The electric field $\underline{E}$ at C due to the remaining charges can then be found using $\underline{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_1^4 \dfrac{q_i \cdot \underline{r}_i}{|\underline{r}_i|^3}$ where q_1 to q_4 are the equal charges (10^{-9} coulombs) and $\underline{r}_1$ to $\underline{r}_4$ are the vectors directed from the four corners, where the charges are located, towards C. In this case since $q_1 = 10^{-9}$ coulombs and $|\underline{r}_i| = \sqrt{1.5}$ for $i = 1$ to $i = 4$ we have

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{10^{-9}}{(1.5)^{3/2}}\left[\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4\right],$$

where $\underline{r}_1 = \underline{AC} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\underline{r}_2 = \underline{BC} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ etc.

Thus $\quad \underline{E} = \dfrac{1}{4\pi\varepsilon_0}\ \dfrac{10^{-9}}{(1.5)^{3/2}}\ \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$

and $\quad |\underline{E}| = \dfrac{1}{\pi\varepsilon_0}\ \dfrac{10^{-9}}{(1.5)^{3/2}} = \dfrac{10^{-9}}{\pi \times 8.854 \times 10^{-12}(1.5)^{3/2}} = 19.57$ V m^{-1}

Answer

(c) Suppose the chosen edge to be used connects $A(0,0,1)$ to $B(1,0,1)$ then the centre point (G) will be located at $G\left(\frac{1}{2},0,1\right)$.

By symmetry the field at G due to the charges at A and B will be zero.

We note that the distances $DG,\ OG,\ PG$ and SG are all equal. In the case of OG we calculate by Pythagoras that this distance is $\sqrt{\left(\frac{1}{2}\right)^2 + 0^2 + 1^2} = \sqrt{1.25}$.

Similarly the distances TG and RG are equal to $\sqrt{2.25}$.

Using the result that $\underline{E} = \dfrac{1}{4\pi\varepsilon_0}\ \sum \dfrac{q_i\underline{r}_i}{|\underline{r}_i|^3}$ gives

$$\underline{E} = \frac{10^{-9}}{4\pi\varepsilon_0}\left[\frac{1}{(1.25)^{3/2}}\left\{\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}\right\}\right.$$

$$\left. + \frac{1}{(2.25)^{3/2}}\left\{\begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}\right\}\right]$$

$$= \frac{10^{-9}}{4\pi\varepsilon_0}\left[\frac{1}{(1.25)^{3/2}}\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} + \frac{1}{(2.25)^{3/2}}\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}\right]$$

$$= \frac{10^{-9}}{4\pi\varepsilon_0}\begin{bmatrix} 0 \\ -2.02367 \\ 2.02367 \end{bmatrix}$$

Thus $|\underline{E}| = \dfrac{10^{-9}}{4 \times \pi \times 8.854 \times 10^{-12}}\ \sqrt{0^2 + (-2.02367)^2 + (2.02367)^2}$

$= 25.72$ V m^{-1} (2 d.p.).

Task

If $\underline{E} = -50\underline{i} - 50\underline{j} + 30\underline{k}$ V m^{-1} where $\underline{i},\ \underline{j}$ and $\underline{k}$ are unit vectors in the x, y and z directions respectively, find the differential amount of work done in moving a 2μC point charge a distance of 5 mm.

(a) From $P(1, 2, 3)$ towards $Q(2, 4, 1)$

(b) From $Q(2, 4, 1)$ towards $P(1, 2, 3)$

Your solution

Answer

(a) The work done in moving a 2μC charge through a distance of 5 mm towards Q is

$$\begin{aligned} W = -q\underline{E}.\underline{ds} &= -(2\times 10^{-6})(5\times 10^{-3})\underline{E}.\frac{\underline{PQ}}{|\underline{PQ}|} \\ &= -10^{-8}(-50\underline{i} - 50\underline{j} + 30\underline{k}) \cdot \frac{(\underline{i} + 2\underline{j} - 2\underline{k})}{\sqrt{1^2 + 2^2 + (-2)^2}} \\ &= \frac{10^{-8}(50 + 100 + 60)}{3} = 7\times 10^{-7}J \end{aligned}$$

(b) A similar calculation yields that the work done in moving the same charge through the same distance in the direction from Q to P is $W = -7\times 10^{-7}J$

The Vector Product 9.4

Introduction

In this Section we describe how to find the **vector product** of two vectors. Like the scalar product, its definition may seem strange when first met but the definition is chosen because of its many applications. When vectors are multiplied using the vector product the result is always a vector.

Prerequisites

Before starting this Section you should ...

- know that a vector can be represented as a directed line segment
- know how to express a vector in Cartesian form
- know how to evaluate 3×3 determinants

Learning Outcomes

On completion you should be able to ...

- use the right-handed screw rule
- calculate the vector product of two given vectors
- use determinants to calculate the vector product of two vectors given in Cartesian form

1. The right-handed screw rule

To understand how the vector product is formed it is helpful to consider first the right-handed screw rule. Consider the two vectors $\underline{a}$ and $\underline{b}$ shown in Figure 37.

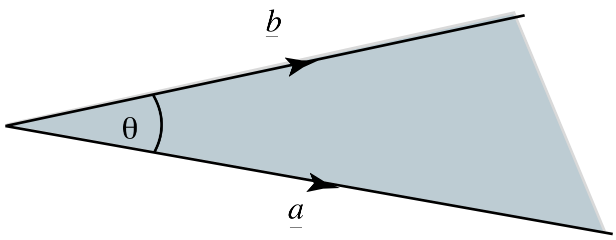


Figure 37

The two vectors lie in a plane; this plane is shaded in Figure 37. Figure 38 shows the same two vectors and the plane in which they lie together with a unit vector, denoted $\hat{\underline{e}}$, which is perpendicular to this plane. Imagine turning a right-handed screw, aligned along $\hat{\underline{e}}$, in the direction from $\underline{a}$ towards $\underline{b}$ as shown. A right-handed screw is one which when turned clockwise enters the material into which it is being screwed (most screws are of this kind). You will see from Figure 38 that the screw will advance in the direction of $\hat{\underline{e}}$.

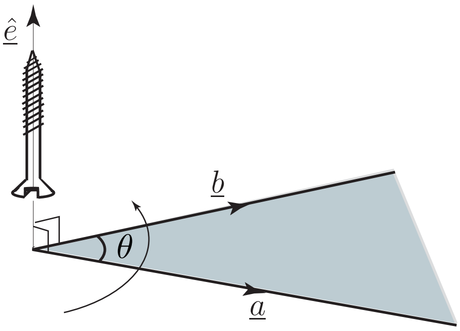


Figure 38

On the other hand, if the right-handed screw is turned from $\underline{b}$ towards $\underline{a}$ the screw will retract in the direction of $\hat{\underline{f}}$ as shown in Figure 39.

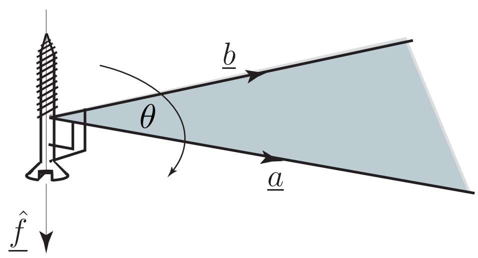


Figure 39

We are now in a position to describe the vector product.

2. Definition of the vector product

We define the vector product of $\underline{a}$ and $\underline{b}$, written $\underline{a} \times \underline{b}$, as

$$\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}|\,\sin\theta\,\hat{\underline{e}}$$

By inspection of this formula note that this is a **vector** of magnitude $|\underline{a}|\,|\underline{b}|\,\sin\theta$ in the direction of the vector $\hat{\underline{e}}$, where $\hat{\underline{e}}$ is a unit vector perpendicular to the plane containing $\underline{a}$ and $\underline{b}$ in the sense defined by the right-handed screw rule. The quantity $\underline{a} \times \underline{b}$ is read as "$\underline{a}$ cross $\underline{b}$" and is sometimes referred to as the **cross product**. The angle is chosen to lie between 0 and π. See Figure 40.

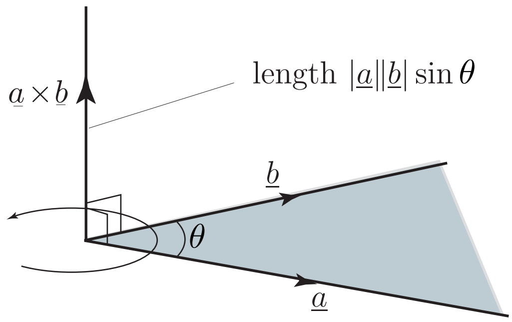


Figure 40: $\underline{a} \times \underline{b}$ is perpendicular to the plane containing $\underline{a}$ and $\underline{b}$

Formally we have

Key Point 17

The vector product of $\underline{a}$ and $\underline{b}$ is: $\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}|\,\sin\theta\,\hat{\underline{e}}$

The modulus of the vector product is: $|\underline{a} \times \underline{b}| = |\underline{a}|\,|\underline{b}|\,\sin\theta$

Note that $|\underline{a}|\,|\underline{b}|\,\sin\theta$ gives the modulus of the vector product whereas $\hat{\underline{e}}$ gives its direction.
Now study Figure 41 which is used to illustrate the calculation of $\underline{b} \times \underline{a}$. In particular note the direction of $\underline{b} \times \underline{a}$ arising through the application of the right-handed screw rule.
We see that $\underline{a} \times \underline{b}$ is not equal to $\underline{b} \times \underline{a}$ because their directions are **opposite**. In fact $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$.

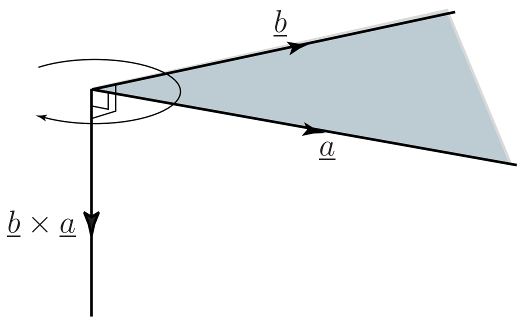


Figure 41: $\underline{b} \times \underline{a}$ is in the opposite direction to $\underline{a} \times \underline{b}$

Example 15

If $\underline{a}$ and $\underline{b}$ are parallel, show that $\underline{a} \times \underline{b} = \underline{0}$.

Solution

If $\underline{a}$ and $\underline{b}$ are parallel then the angle θ between them is zero. Consequently $\sin\theta = 0$ from which it follows that $\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}|\,\sin\theta\,\hat{\underline{e}} = \underline{0}$. Note that the result, $\underline{0}$, is the **zero vector**.

Note in particular the important results which follow:

Key Point 18

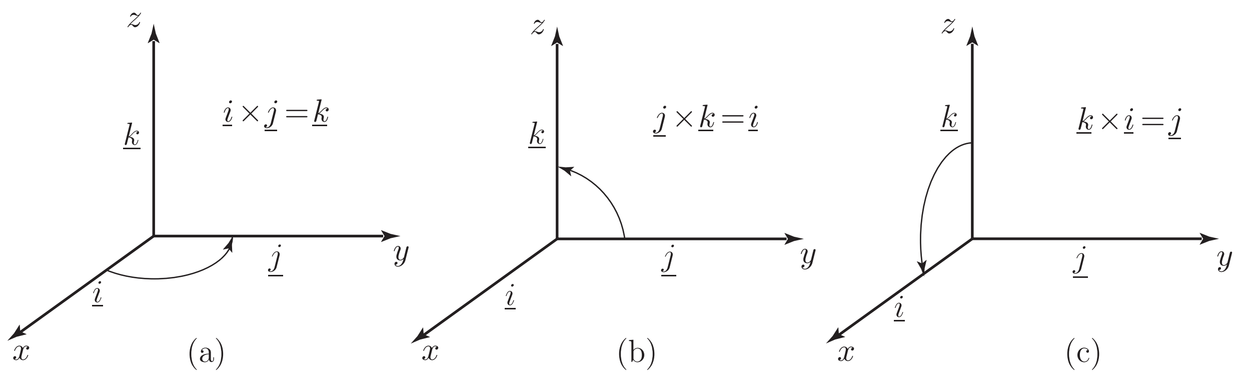
$$\underline{i} \times \underline{i} = \underline{0} \qquad \underline{j} \times \underline{j} = \underline{0} \qquad \underline{k} \times \underline{k} = \underline{0}$$

Example 16

Show that $\underline{i} \times \underline{j} = \underline{k}$ and find expressions for $\underline{j} \times \underline{k}$ and $\underline{k} \times \underline{i}$.

Solution

Note that $\underline{i}$ and $\underline{j}$ are perpendicular so that the angle between them is 90°. Also, the vector $\underline{k}$ is perpendicular to both $\underline{i}$ and $\underline{j}$. Using Key Point 17, the modulus of $\underline{i} \times \underline{j}$ is $(1)(1)\sin 90^\circ = 1$. So $\underline{i}\times\underline{j}$ is a unit vector.The unit vector perpendicular to $\underline{i}$ and $\underline{j}$ in the sense defined by the right-handed screw rule is $\underline{k}$ as shown in Figure 42(a). Therefore $\underline{i} \times \underline{j} = \underline{k}$ as required.



Similarly you should verify that $\underline{j} \times \underline{k} = \underline{i}$ (Figure 42(b)) and $\underline{k} \times \underline{i} = \underline{j}$ (Figure 42(c)).

Key Point 19

$\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j} \quad \underline{j} \times \underline{i} = -\underline{k}, \quad \underline{k} \times \underline{j} = -\underline{i}, \quad \underline{i} \times \underline{k} = -\underline{j}$

To help remember these results you might like to think of the vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ written in alphabetical order like this:

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{i} \quad \underline{j} \quad \underline{k}$$

Moving left to right yields a positive result: e.g. $\underline{k} \times \underline{i} = \underline{j}$.

Moving right to left yields a negative result: e.g. $\underline{j} \times \underline{i} = -\underline{k}$.

3. A formula for finding the vector product

We can use the results in Key Point 19 to develop a formula for finding the vector product of two vectors given in Cartesian form: Suppose $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\begin{aligned}
\underline{a} \times \underline{b} &= (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\
&= a_1\underline{i} \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\
&\qquad + \quad a_2\underline{j} \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\
&\qquad\qquad + \quad a_3\underline{k} \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\
&= a_1b_1(\underline{i} \times \underline{i}) + a_1b_2(\underline{i} \times \underline{j}) + a_1b_3(\underline{i} \times \underline{k}) \\
&\qquad + \quad a_2b_1(\underline{j} \times \underline{i}) + a_2b_2(\underline{j} \times \underline{j}) + a_2b_3(\underline{j} \times \underline{k}) \\
&\qquad\qquad + \quad a_3b_1(\underline{k} \times \underline{i}) + a_3b_2(\underline{k} \times \underline{j}) + a_3b_3(\underline{k} \times \underline{k})
\end{aligned}$$

Using Key Point 19, this expression simplifies to

$$\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

This gives us Key Point 20:

Key Point 20

If $\quad \underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k} \quad$ and $\quad \underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k} \quad$ then

$$\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

Example 17

Evaluate the vector product $\underline{a} \times \underline{b}$ if $\underline{a} = 3\underline{i} - 2\underline{j} + 5\underline{k}$ and $\underline{b} = 7\underline{i} + 4\underline{j} - 8\underline{k}$.

Solution

Identifying $a_1 = 3$, $a_2 = -2$, $a_3 = 5$, $b_1 = 7$, $b_2 = 4$, $b_3 = -8$ we find

$$\begin{aligned}\underline{a} \times \underline{b} &= ((-2)(-8) - (5)(4))\underline{i} - ((3)(-8) - (5)(7))\underline{j} + ((3)(4) - (-2)(7))\underline{k} \\ &= -4\underline{i} + 59\underline{j} + 26\underline{k}\end{aligned}$$

Task

Use Key Point 20 to find the vector product of $\underline{p} = 3\underline{i} + 5\underline{j}$ and $\underline{q} = 2\underline{i} - \underline{j}$.

Note that in this case there are no $\underline{k}$ components so a_3 and b_3 are both zero:

Your solution

$\underline{p} \times \underline{q} =$

Answer

$-13\underline{k}$

4. Using determinants to evaluate a vector product

Evaluation of a vector product using the formula in Key Point 20 is very cumbersome. A more convenient and easily remembered method is to use determinants. Recall from Workbook 7 that, for a 3×3 determinant,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The vector product of two vectors $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ can be found by evaluating the determinant:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

in which $\underline{i}$, $\underline{j}$ and $\underline{k}$ are (temporarily) treated as if they were scalars.

Key Point 21

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}(a_2b_3 - a_3b_2) - \underline{j}(a_1b_3 - a_3b_1) + \underline{k}(a_1b_2 - a_2b_1)$$

Example 18

Find the vector product of $\underline{a} = 3\underline{i} - 4\underline{j} + 2\underline{k}$ and $\underline{b} = 9\underline{i} - 6\underline{j} + 2\underline{k}$.

Solution

We have $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 2 \\ 9 & -6 & 2 \end{vmatrix}$ which, when evaluated, gives

$$\underline{a} \times \underline{b} = \underline{i}(-8 - (-12)) - \underline{j}(6 - 18) + \underline{k}(-18 - (-36)) = 4\underline{i} + 12\underline{j} + 18\underline{k}$$

Example 19

The area A_T of the triangle shown in Figure 43 is given by the formula $A_T = \frac{1}{2}bc\sin\alpha$. Show that an equivalent formula is $A_T = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

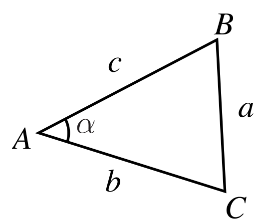


Figure 43

Solution

We use the definition of the vector product $|\overrightarrow{AB} \times \overrightarrow{AC}| = |\overrightarrow{AB}|\,|\overrightarrow{AC}|\sin\alpha$.

Since α is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and $|\overrightarrow{AB}| = c$ and $|\overrightarrow{AC}| = b$, the required result follows immediately:

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}c \cdot b \cdot \sin\alpha.$$

Moments

The **moment** (or **torque**) of the force $\underline{F}$ about a point O is defined as

$$\underline{M}_o = \underline{r} \times \underline{F}$$

where $\underline{r}$ is a position vector from O to *any* point on the line of action of $\underline{F}$ as shown in Figure 44.

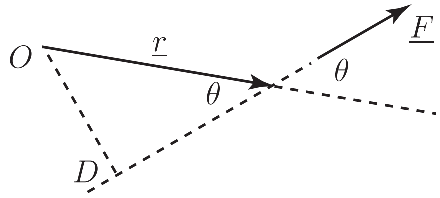


Figure 44

It may seem strange that any point on the line of action may be taken but it is easy to show that exactly the same vector $\underline{M}_o$ is always obtained.
By the properties of the cross product the direction of $\underline{M}_o$ is perpendicular to the plane containing $\underline{r}$ and $\underline{F}$ (i.e. out of the paper). The magnitude of the moment is

$$|\underline{M}_0| = |\underline{r}||\underline{F}|\sin\theta.$$

From Figure 32, $|\underline{r}|\sin\theta = D$. Hence $|\underline{M}_o| = D|\underline{F}|$. This would be the same no matter which point on the line of action of $\underline{F}$ was chosen.

Example 20

Find the moment of the force given by $\underline{F} = 3\underline{i} + 4\underline{j} + 5\underline{k}$ (N) acting at the point $(14, -3, 6)$ about the point $P(2, -2, 1)$.

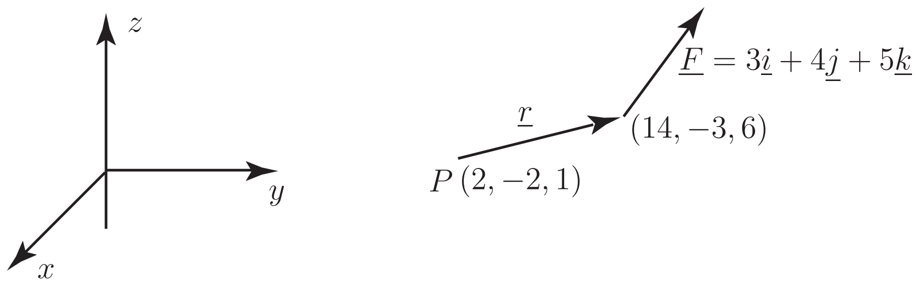


Figure 45

Solution

The vector $\underline{r}$ can be any vector from the point P to any point on the line of action of $\underline{F}$. Choosing $\underline{r}$ to be the vector connecting P to $(14, -3, 6)$ (and measuring distances in metres) we have:

$$\underline{r} = (14-2)\underline{i} + (-3-(-2))\underline{j} + (6-1)\underline{k} = 12\underline{i} - \underline{j} + 5\underline{k}.$$

The moment is $\quad \underline{M} = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 12 & -1 & 5 \\ 3 & 4 & 5 \end{vmatrix} = -25\underline{i} - 45\underline{j} + 51\underline{k}$ (N m)

Exercises

1. Show that if $\underline{a}$ and $\underline{b}$ are parallel vectors then their vector product is the zero vector.
2. Find the vector product of $\underline{p} = -2\underline{i} - 3\underline{j}$ and $\underline{q} = 4\underline{i} + 7\underline{j}$.
3. If $\underline{a} = \underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{b} = 4\underline{i} + 3\underline{j} + 2\underline{k}$ find $\underline{a} \times \underline{b}$. Show that $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$.
4. Points A, B and C have coordinates $(9, 1, -2)$, (3,1,3), and $(1, 0, -1)$ respectively. Find the vector product $\overrightarrow{AB} \times \overrightarrow{AC}$.
5. Find a vector which is perpendicular to both of the vectors $\underline{a} = \underline{i} + 2\underline{j} + 7\underline{k}$ and $\underline{b} = \underline{i} + \underline{j} - 2\underline{k}$. Hence find a unit vector which is perpendicular to both $\underline{a}$ and $\underline{b}$.
6. Find a vector which is perpendicular to the plane containing $6\underline{i} + \underline{k}$ and $2\underline{i} + \underline{j}$.
7. For the vectors $\underline{a} = 4\underline{i} + 2\underline{j} + \underline{k}$, $\underline{b} = \underline{i} - 2\underline{j} + \underline{k}$, and $\underline{c} = 3\underline{i} - 3\underline{j} + 4\underline{k}$, evaluate both $\underline{a} \times (\underline{b} \times \underline{c})$ and $(\underline{a} \times \underline{b}) \times \underline{c}$. Deduce that, in general, the vector product is not associative.
8. Find the area of the triangle with vertices at the points with coordinates $(1, 2, 3)$, $(4, -3, 2)$ and $(8, 1, 1)$.
9. For the vectors $\underline{r} = \underline{i} + 2\underline{j} + 3\underline{k}$, $\underline{s} = 2\underline{i} - 2\underline{j} - 5\underline{k}$, and $\underline{t} = \underline{i} - 3\underline{j} - \underline{k}$, evaluate
   (a) $(\underline{r} \cdot \underline{t})\underline{s} - (\underline{s} \cdot \underline{t})\underline{r}$. (b) $(\underline{r} \times \underline{s}) \times \underline{t}$. Deduce that $(\underline{r} \cdot \underline{t})\underline{s} - (\underline{s} \cdot \underline{t})\underline{r} = (\underline{r} \times \underline{s}) \times \underline{t}$.

Answers

1. This uses the fact that $\sin 0 = 0$.
2. $-2\underline{k}$
3. $-5\underline{i} + 10\underline{j} - 5\underline{k}$
4. $5\underline{i} - 34\underline{j} + 6\underline{k}$
5. $-11\underline{i} + 9\underline{j} - \underline{k}$, $\quad \dfrac{1}{\sqrt{203}}(-11\underline{i} + 9\underline{j} - \underline{k})$
6. $-\underline{i} + 2\underline{j} + 6\underline{k}$ for example.
7. $7\underline{i} - 17\underline{j} + 6\underline{k}$, $\quad -42\underline{i} - 46\underline{j} - 3\underline{k}$. These are different so the vector product is **not** associative.
8. $\frac{1}{2}\sqrt{1106}$
9. Each gives $-29\underline{i} - 10\underline{j} + \underline{k}$

Lines and Planes

9.5

Introduction

Vectors are very convenient tools for analysing lines and planes in three dimensions. In this Section you will learn about direction ratios and direction cosines and then how to formulate the vector equation of a line and the vector equation of a plane. Direction ratios provide a convenient way of specifying the direction of a line in three dimensional space. Direction cosines are the cosines of the angles between a line and the coordinate axes. We begin this Section by showing how these quantities are calculated.

Prerequisites

Before starting this Section you should ...

- understand and be able to calculate the scalar product of two vectors
- understand and be able to calculate the vector product of two vectors

Learning Outcomes

On completion you should be able to ...

- obtain the vector equation of a line
- obtain the vector equation of a plane passing through a given point and which is perpendicular to a given vector
- obtain the vector equation of a plane which is a given distance from the origin and which is perpendicular to a given vector

1. The direction ratio and direction cosines

Consider the point $P(4, 5)$ and its position vector $4\underline{i} + 5\underline{j}$ shown in Figure 46.

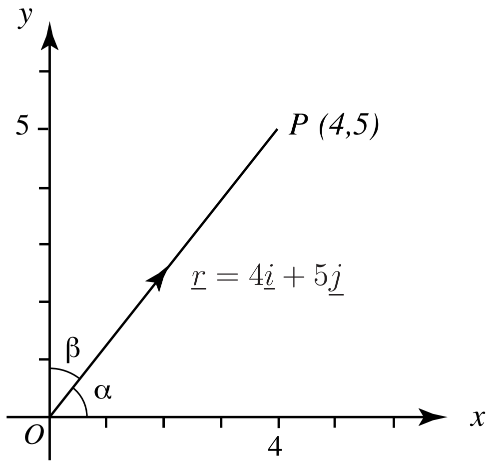


Figure 46

The **direction ratio** of the vector $\overrightarrow{OP}$ is defined to be 4:5. We can interpret this as stating that to move in the direction of the line OP we must move 4 units in the x direction for every 5 units in the y direction.

The **direction cosines** of the vector $\overrightarrow{OP}$ are the cosines of the angles between the vector and each of the axes. Specifically, referring to Figure 46 these are

$\cos\alpha$ and $\cos\beta$

Noting that the length of $\overrightarrow{OP}$ is $\sqrt{4^2 + 5^2} = \sqrt{41}$, we can write

$$\cos\alpha = \frac{4}{\sqrt{41}}, \qquad \cos\beta = \frac{5}{\sqrt{41}}.$$

It is conventional to label the direction cosines as ℓ and m so that

$$\ell = \frac{4}{\sqrt{41}}, \qquad m = \frac{5}{\sqrt{41}}.$$

More generally we have the following result:

Key Point 22

For **any** vector $\underline{r} = a\underline{i} + b\underline{j}$, its direction ratio is $a : b$.

Its direction cosines are

$$\ell = \frac{a}{\sqrt{a^2 + b^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2}}$$

Example 21

Point A has coordinates $(3, 5)$, and point B has coordinates $(7, 8)$.

(a) Write down the vector $\overrightarrow{AB}$.

(b) Find the direction ratio of the vector $\overrightarrow{AB}$.

(c) Find its direction cosines, ℓ and m.

(d) Show that $\ell^2 + m^2 = 1$.

Solution

(a) $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$.

(b) The direction ratio of $\overrightarrow{AB}$ is therefore 4:3.

(c) The direction cosines are

$$\ell = \frac{4}{\sqrt{4^2 + 3^2}} = \frac{4}{5}, \qquad m = \frac{3}{\sqrt{4^2 + 3^2}} = \frac{3}{5}$$

(d)

$$\ell^2 + m^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

The final result in the previous Example is true in general:

Key Point 23

If ℓ and m are the direction cosines of a vector lying in the xy plane, then $\ell^2 + m^2 = 1$

Exercise

P and Q have coordinates $(-2, 4)$ and $(7, 8)$ respectively.

(a) Find the direction ratio of the vector $\overrightarrow{PQ}$ (b) Find the direction cosines of $\overrightarrow{PQ}$.

Answer

(a) $9 : 4$, (b) $\frac{9}{\sqrt{97}}, \frac{4}{\sqrt{97}}$.

2. Direction ratios and cosines in three dimensions

The concepts of direction ratio and direction cosines extend naturally to three dimensions. Consider Figure 47.

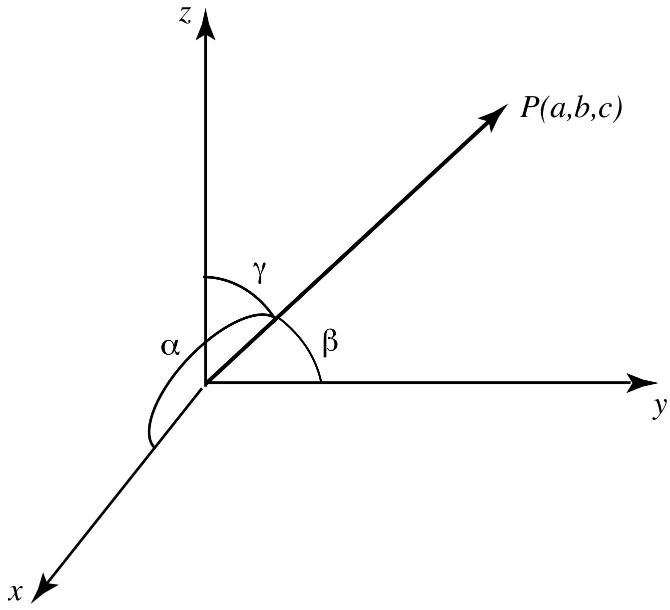


Figure 47

Given a vector $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ its direction ratios are $a : b : c$. This means that to move in the direction of the vector we must must move a units in the x direction and b units in the y direction for every c units in the z direction.

The direction cosines are the cosines of the angles between the vector and each of the axes. It is conventional to label direction cosines as ℓ, m and n and they are given by

$$\ell = \cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Wee have the following general result:

Key Point 24

For any vector $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ its direction ratios are $a : b : c$.

Its direction cosines are

$$\ell = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

where $\ell^2 + m^2 + n^2 = 1$

Exercises

1. Points A and B have position vectors $\underline{a} = -3\underline{i} + 2\underline{j} + 7\underline{k}$, and $\underline{b} = 3\underline{i} + 4\underline{j} - 5\underline{k}$ respectively. Find

   (a) $\overrightarrow{AB}$

   (b) $|\overrightarrow{AB}|$

   (c) The direction ratios of $\overrightarrow{AB}$

   (d) The direction cosines (ℓ, m, n) of $\overrightarrow{AB}$.

   (e) Show that $\ell^2 + m^2 + n^2 = 1$.

2. Find the direction ratios, the direction cosines and the angles that the vector $\overrightarrow{OP}$ makes with each of the axes when P is the point with coordinates $(2, 4, 3)$.

3. A line is inclined at $60°$ to the x axis and $45°$ to the y axis. Find its inclination to the z axis.

Answers

1. (a) $6\underline{i} + 2\underline{j} - 12\underline{k}$, (b) $\sqrt{184}$, (c) $6 : 2 : -12$, (d) $\dfrac{6}{\sqrt{184}}, \dfrac{2}{\sqrt{184}}, \dfrac{-12}{\sqrt{184}}$

2. 2:4:3; $\dfrac{2}{\sqrt{29}}, \dfrac{4}{\sqrt{29}}, \dfrac{3}{\sqrt{29}}$; $68.2°$, $42.0°$, $56.1°$.

3. $60°$ or $120°$.

3. The vector equation of a line

Consider the straight line APB shown in Figure 48. This is a line in three-dimensional space.

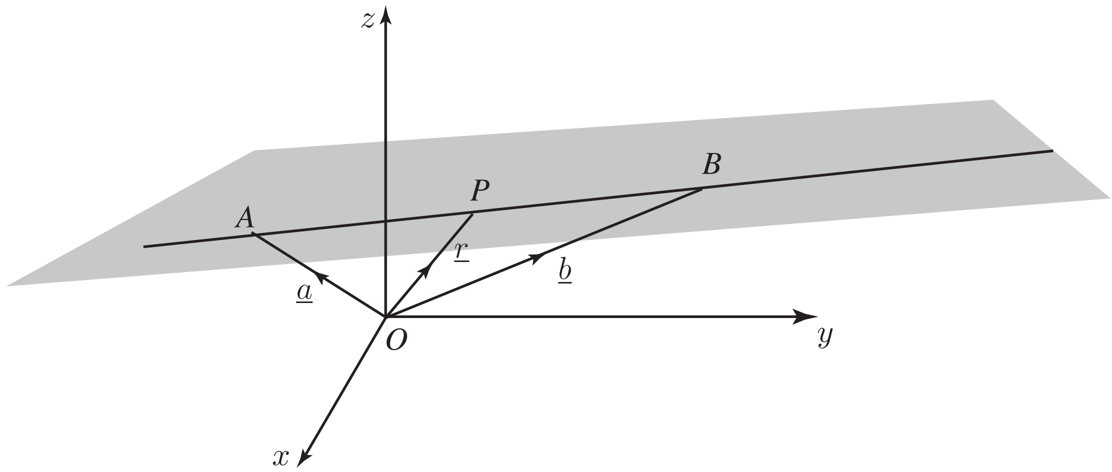


Figure 48

Points A and B are fixed and known points on the line, and have position vectors $\underline{a}$ and $\underline{b}$ respectively. Point P is any other arbitrary point on the line, and has position vector $\underline{r}$. Note that because $\overrightarrow{AB}$ and $\overrightarrow{AP}$ are parallel, $\overrightarrow{AP}$ is simply a scalar multiple of $\overrightarrow{AB}$, that is, $\overrightarrow{AP} = t\overrightarrow{AB}$ where t is a number.

Task

Referring to Figure 48, write down an expression for the vector $\overrightarrow{AB}$ in terms of $\underline{a}$ and $\underline{b}$.

Your solution

Answer

$\overrightarrow{AB} = \underline{b} - \underline{a}$

Task

Referring to Figure 48, use the triangle law for vector addition to find an expression for $\underline{r}$ in terms of $\underline{a}$, $\underline{b}$ and t, where $\overrightarrow{AP} = t\overrightarrow{AB}$.

Your solution

Answer

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

so that

$$\underline{r} = \underline{a} + t(\underline{b} - \underline{a}) \qquad \text{since } \overrightarrow{AP} = t\overrightarrow{AB}$$

The answer to the above Task, $\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$, is the **vector equation of the line** through A and B. It is a rule which gives the position vector $\underline{r}$ of a general point on the line in terms of the **given** vectors $\underline{a}, \underline{b}$. By varying the value of t we can move to any point on the line. For example, referring to Figure 48,

when $t = 0$, the equation gives $\underline{r} = \underline{a}$, which locates point A,

when $t = 1$, the equation gives $\underline{r} = \underline{b}$, which locates point B.

If $0 < t < 1$ the point P lies on the line between A and B. If $t > 1$ the point P lies on the line beyond B (to the right in the figure). If $t < 0$ the point P lies on the line beyond A (to the left in the figure).

Key Point 25

The **vector equation of the line** through points A and B with position vectors $\underline{a}$ and $\underline{b}$ is

$$\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$$

Task

Write down the vector equation of the line which passes through the points with position vectors $\underline{a} = 3\underline{i} + 2\underline{j}$ and $\underline{b} = 7\underline{i} + 5\underline{j}$. Also express the equation in column vector form.

Your solution

Answer

$$\underline{b} - \underline{a} = (7\underline{i} + 5\underline{j}) - (3\underline{i} + 2\underline{j}) = 4\underline{i} + 3\underline{j}$$

The equation of the line is then

$$\begin{aligned} \underline{r} &= \underline{a} + t(\underline{b} - \underline{a}) \\ &= (3\underline{i} + 2\underline{j}) + t(4\underline{i} + 3\underline{j}) \end{aligned}$$

Using column vector notation we could write

$$\underline{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Task

Using column vector notation, write down the vector equation of the line which passes through the points with position vectors $\underline{a} = 5\underline{i} - 2\underline{j} + 3\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} - 4\underline{k}$.

Your solution

Answer

Using column vector notation note that $\underline{b} - \underline{a} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -7 \end{pmatrix}$

The equation of the line is then $\underline{r} = \underline{a} + t(\underline{b} - \underline{a}) = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -3 \\ 3 \\ -7 \end{pmatrix}$

Cartesian form

On occasions it is useful to convert the vector form of the equation of a straight line into Cartesian form. Suppose we write

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

then $\underline{r} = \underline{a} + t(\underline{b} - \underline{a})$ implies

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} = \begin{pmatrix} a_1 + t(b_1 - a_1) \\ a_2 + t(b_2 - a_2) \\ a_3 + t(b_3 - a_3) \end{pmatrix}$$

Equating the individual components we find

$$x = a_1 + t(b_1 - a_1), \qquad \text{or equivalently } t = \frac{x - a_1}{b_1 - a_1}$$

$$y = a_2 + t(b_2 - a_2), \qquad \text{or equivalently } t = \frac{y - a_2}{b_2 - a_2}$$

$$z = a_3 + t(b_3 - a_3), \qquad \text{or equivalently } t = \frac{z - a_3}{b_3 - a_3}$$

Each expression on the right is equal to t and so we can write

$$\frac{x-a_1}{b_1-a_1}=\frac{y-a_2}{b_2-a_2}=\frac{z-a_3}{b_3-a_3}$$

This gives the **Cartesian form** of the equations of the straight line which passes through the points with coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3).

Key Point 26

The **Cartesian form** of the equation of the straight line which passes through the points with coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) is

$$\frac{x-a_1}{b_1-a_1}=\frac{y-a_2}{b_2-a_2}=\frac{z-a_3}{b_3-a_3}$$

Example 22

(a) Write down the Cartesian form of the equation of the straight line which passes through the two points $(9, 3, -2)$ and $(4, 5, -1)$.

(b) State the equivalent vector equation.

Solution

(a)

$$\frac{x-9}{4-9}=\frac{y-3}{5-3}=\frac{z-(-2)}{-1-(-2)}$$

that is

$$\frac{x-9}{-5}=\frac{y-3}{2}=\frac{z+2}{1} \qquad \text{(Cartesian form)}$$

(b) The vector equation is

$$\begin{aligned}\underline{r} &= \underline{a}+t(\underline{b}-\underline{a})\\ &= \begin{pmatrix}9\\3\\-2\end{pmatrix}+t(\begin{pmatrix}4\\5\\-1\end{pmatrix}-\begin{pmatrix}9\\3\\-2\end{pmatrix})\\ &= \begin{pmatrix}9\\3\\-2\end{pmatrix}+t\begin{pmatrix}-5\\2\\1\end{pmatrix}\end{aligned}$$

Exercises

1. (a) Write down the vector $\overrightarrow{AB}$ joining the points A and B with coordinates $(3, 2, 7)$ and $(-1, 2, 3)$ respectively.

   (b) Find the equation of the straight line through A and B.

2. Write down the vector equation of the line passing through the points with position vectors $\underline{p} = 3\underline{i} + 7\underline{j} - 2\underline{k}$ and $\underline{q} = -3\underline{i} + 2\underline{j} + 2\underline{k}$. Find also the Cartesian equation of this line.

3. Find the vector equation of the line passing through $(9, 1, 2)$ and which is parallel to the vector $(1, 1, 1)$.

Answers

1. (a) $-4\underline{i} - 4\underline{k}$. (b) $\underline{r} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ -4 \end{pmatrix}$.

2. $\underline{r} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix} + t\begin{pmatrix} -6 \\ -5 \\ 4 \end{pmatrix}$. Cartesian form $\dfrac{x-3}{-6} = \dfrac{y-7}{-5} = \dfrac{z+2}{4}$.

3. $\underline{r} = \begin{pmatrix} 9 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

4. The vector equation of a plane

Consider the plane shown in Figure 49.

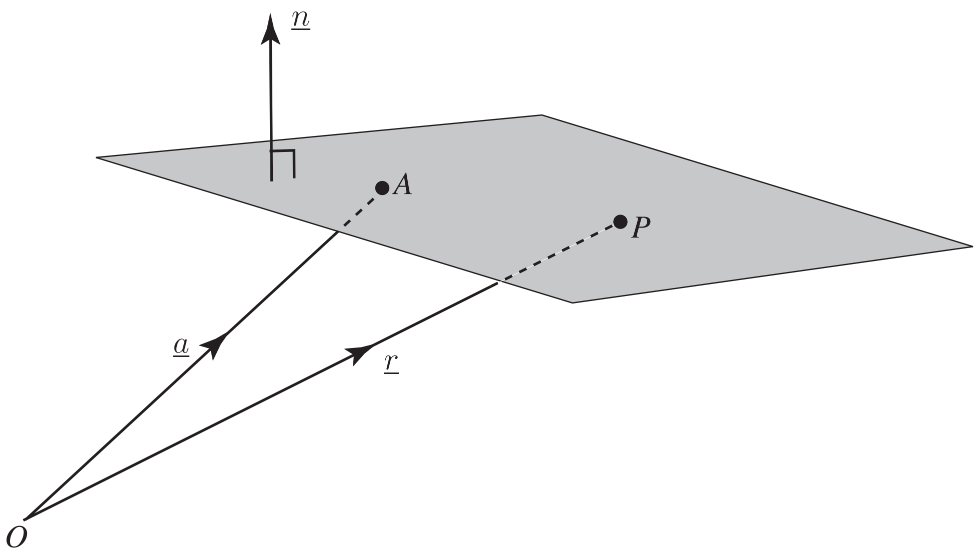


Figure 49

Suppose that A is a fixed point in the plane and has position vector $\underline{a}$. Suppose that P is any other arbitrary point in the plane with position vector $\underline{r}$. Clearly the vector $\overrightarrow{AP}$ lies in the plane.

Task

Referring to Figure 49, find the vector $\overrightarrow{AP}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{r}$.

Your solution

Answer

$\underset{\sim}{r} - \underset{\sim}{a}$

Also shown in Figure 49 is a vector which is perpendicular to the plane and denoted by $\underset{\sim}{n}$.

Task

What relationship exists between $\underset{\sim}{n}$ and the vector $\overrightarrow{AP}$?

Hint: think about the scalar product:

Your solution

Answer

Because $\overrightarrow{AP}$ and $\underset{\sim}{n}$ are perpendicular their scalar product must equal zero, that is

$$(\underset{\sim}{r} - \underset{\sim}{a}).\underset{\sim}{n} = 0 \qquad \text{so that} \qquad \underset{\sim}{r}.\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$$

The answer to the above Task, $\underset{\sim}{r}.\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$, is the **equation of a plane**, written in vector form, passing through A and perpendicular to $\underset{\sim}{n}$.

Key Point 27

A plane perpendicular to the vector $\underset{\sim}{n}$ and passing through the point with position vector $\underset{\sim}{a}$, has equation

$$\underset{\sim}{r}.\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$$

In this formula it does not matter whether or not $\underline{n}$ is a unit vector.

If $\hat{\underline{n}}$ is a unit vector then $\underline{a}.\hat{\underline{n}}$ represents the perpendicular distance from the origin to the plane which we usually denote by d (for details of this see Section 9.3). Hence we can write

$$\underline{r}.\hat{\underline{n}} = d$$

This is the **equation of a plane**, written in vector form, with unit normal $\hat{\underline{n}}$ and which is a perpendicular distance d from O.

Key Point 28

A plane with unit normal $\hat{\underline{n}}$, which is a perpendicular distance d from O is given by

$$\underline{r}.\hat{\underline{n}} = d$$

Example 23

(a) Find the vector equation of the plane which passes through the point with position vector $3\underline{i} + 2\underline{j} + 5\underline{k}$ and which is perpendicular to $\underline{i} + \underline{k}$.

(b) Find the Cartesian equation of this plane.

Solution

(a) Using the previous results we can write down the equation

$$\underline{r}.(\underline{i} + \underline{k}) = (3\underline{i} + 2\underline{j} + 5\underline{k}).(\underline{i} + \underline{k}) = 3 + 5 = 8$$

(b) Writing $\underline{r}$ as $x\underline{i} + y\underline{j} + z\underline{k}$ we have the Cartesian form:

$$(x\underline{i} + y\underline{j} + z\underline{k}).(\underline{i} + \underline{k}) = 8$$

so that

$$x + z = 8$$

Task

(a) Find the vector equation of the plane through $(7, 3, -5)$ for which $\underline{n} = (1, 1, 1)$ is a vector normal to the plane.
(b) What is the distance of the plane from O?

Your solution

Answer

(a) Using the formula $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$ the equation of the plane is

$$\underline{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 7 \times 1 + 3 \times 1 - 5 \times 1 = 5$$

(b) The distance from the origin is $\underline{a}.\underline{\hat{n}} = \begin{pmatrix} 7 \\ 3 \\ -5 \end{pmatrix} \cdot \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \dfrac{5}{\sqrt{3}}$

Exercises

1. Find the equation of a plane which is normal to $8\underline{i} + 9\underline{j} + \underline{k}$ and which is a distance 1 from the origin. Give both vector and Cartesian forms.

2. Find the equation of a plane which passes through $(8, 1, 0)$ and which is normal to the vector $\underline{i} + 2\underline{j} - 3\underline{k}$.

3. What is the distance of the plane $\underline{r}. \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 5$ from the origin?

Answers

1. $\underline{r} \cdot \dfrac{1}{\sqrt{146}} \begin{pmatrix} 8 \\ 9 \\ 1 \end{pmatrix} = 1; \quad 8x + 9y + z = \sqrt{146}.$

2. $\underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, that is $\underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 10.$

3. $\dfrac{5}{\sqrt{14}}$